DOCUMENT RESUME

ED 184 828 SE 030 437

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TITLE Analytic Geometry, Student Text, Part 2. Revised Edition.

INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.

SPONS AGENCY National Science Foundation, Washington, D.C.

PUB DATE 64

NOTE 244p.; For related documents, see SE 030 435-440 and ED 143 512-513. Contains occasional light type.

EDRS PRICE MF01/PC10 Plus Postage.

DESCRIPTORS Algebra; *Analytic Geometry; Geometric Concepts; Geometry; Mathematics Curriculum; Mathematics Instruction; Secondary Education; *Secondary School Mathematics; *Textbooks

IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is part two of a three-part School Mathematics Study Group (SMSG) textbook. This part includes the index and chapters 6 through 10: (6) Curve Sketching and Locus Problems, (7) Conic Sections, (8) The Line and the Plane in 3-Space, (9) Quadric Surfaces, and (10) Geometric Transformations. (MK)

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ANALYTIC GEOMETRY

Student Text

Part 2

(revised edition)

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ANALYTIC GEOMETRY

Student Text

Part 2

(revised edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

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Printed in the United States of America

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

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Chapter 6

CURVE SKETCHING AND LOCUS PROBLEMS

6-1. Introduction

We have by this time made a beginning in the discussion of sets of points and their analytic descriptions. We have introduced and used various coordinate systems. We have used parametric representations, finding them particularly useful in physical applications involving rotation or other motion, and in locating positions on a path. Now we investigate more of the details and try to develop more competence (and confidence) in this powerful language of analytic geometry.

6-2. General Principles

The study of analytic geometry has two major concerns. One of these is the relation of geometry to algebra; the other is the relation of algebra to geometry. We must, therefore, consider two basic situations.

A. We are given a set of points. What would be a good analytic representation of that set? If we had two sets of points how would their geometric relationships be revealed in their analytic representations? (geometry to algebra.)

B. We are given an analytic representation of a set of points. What can we now say about the geometric properties of that set? If we had analytic representations of two sets, how could we use these to reveal and develop their geometric relationships? (algebra to geometry.)

In the first situation we must distinguish immediately between the cases we shall treat in this text and those we leave for later work. If a set of points comes to us, say, from a chart of the results of an experiment or a curve drawn by an automatic recording device, it might be useful to have a simple analytic representation of that set. We do not treat such matters in this book, although they have important applications in science, and are the subject of much current mathematical research.

The sets of points with which we shall concern ourselves must come already structured by some geometric condition or property. Our task will be to translate this condition into analytic terms through our choice of coordinate system and mode of algebraic or trigonometric representation. For example we may be interested in the set of all points equidistant from two given points. What type of coordinate system is best suited to describe this situation? Can we simplify the description by a wise choice of axes and units?

On the other hand suppose we meet the expression $2x + 3y + 5 \geq 0$. What set of points does it describe? Is it a configuration we can visualize? What are its properties?

In this second situation the variables come to us already named, and the context and notation usually indicate the type of coordinate system and the choices of axes and units. The analytic representation may exhibit some special algebraic or trigonometric properties which we expect to see reflected in certain geometric properties of the corresponding graph. We do not define the general term, "property", but illustrate and comment on those we shall consider.

Example 1. Discuss the equation $y = \sin x$ and its graph.

Discussion: We assume that the domain of x is the set of real numbers and note immediately that, whatever the value of x, we always have $|y| \leq 1$. If a graph of this equation were drawn on the usual rectangular coordinate grid the geometric interpretation of this statement is that the entire graph is contained in a strip two units wide, centered on the x-axis, and of infinite length to right and left. We sometimes describe such restrictions on the graph by saying it is bounded above and below, but not at the sides. Any comment indicating what regions of the plane may or may not be occupied by a graph is part of the discussion of what is called the extent of the graph.

We note also from the given relationship, that for each value of x there is a unique value of y, but not vice versa. That is, y is expressed as a function of x, but x is not a function of y. The geometric version of this comment is that, if the graph were drawn on the usual rectangular coordinate grid, each line parallel to the y-axis would intersect the graph exactly once. What can you say about intersections of the graph with lines parallel to the x-axis?

We note also that, since $\sin(x + 2n\pi) = \sin x$ for integral values of n, the y values will repeat endlessly through the range $-1 \leq y \leq 1$. We say in this case that y is a periodic function of x. If, in general, $y = f(x)$ so that, for some fixed $p \neq 0$ and for all x, $f(x + p) = f(x)$, then we say that y is a periodic function of x. In that case

$$f(x + 2p) = f((x + p) + p) = f(x + p) = f(x) .$$

Therefore, for such functions, $f(x + np) = f(x)$ for integral values of n. If $p > 0$ and there is no smaller positive number which satisfies the requirement: $f(x + p) = f(x)$ for all x, then we say that $f(x)$ is a periodic function of x, of period p.

Specifically, $y = \sin x$ is a periodic function of x of period 2π. What are the periods of the periodic functions, $y = \cos x$ and $y = \tan x$? Note that it is the function which is periodic, not the graph. A particular function may have quite different looking graphs, depending on our choices of coordinate systems. The periodicity of a function may be more readily seen in some graphs than in others. The graph in Figure 6-1 can be interpreted to give the same information about $y = \sin x$ as is given when we say that y is a bounded periodic function of x of period 2π. What other information about the function can be inferred from the graph?

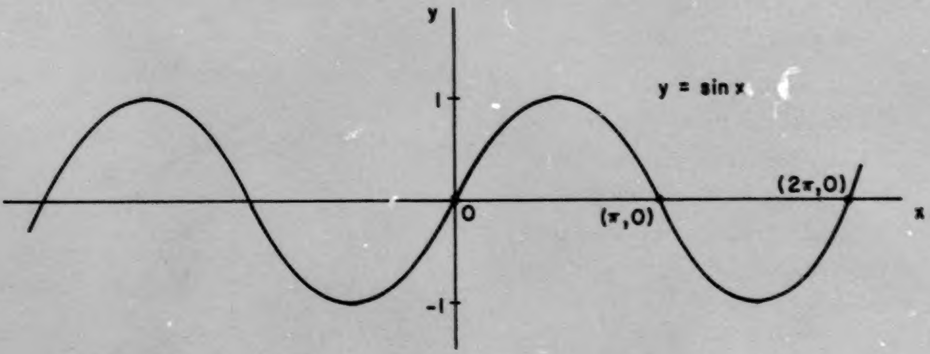


Figure 6-1

We have chosen the usual rectangular coordinate system, using x and y as abscissa and ordinate respectively, and obtained the familiar and beautiful sine curve. Do you see the relation between the shape of this curve and the related words: sinuous, and insinuate?

We could have chosen a polar coordinate system for a graphic representation of $y = \sin x$. We may expect a different looking graph on a different grid, but we should expect also to have some geometric counterparts of the algebraic properties we mentioned earlier.

When we use polar coordinates we customarily use as variables not x and y but r and θ. r is now a measure of the polar distance to the point (r,θ), and θ is a measure of the angle between the polar axis and the polar ray through (r,θ). In this context some authors say that r is a measure of the distance or modulus, and that θ is a measure of the argument, or amplitude.

A strong note of caution must be made in discussions of polar graphs of equations. From the fact that a point does not have a unique representation in polar coordinates we expect that a set of points may have several, perhaps quite dissimilar analytic representations. Any discussion of the relation between a graph and its analytic representation in polar coordinates must take account of this lack of uniqueness. We remember that a point P is on the graph of $r = f(\theta)$ if P has at least one pair of polar coordinates which satisfy this equation. Thus the point $P = (10,5)$ is on the polar graph of $r = 2\theta$, because $10 = 2(5)$, but the same point could also have been located by the coordinates $(10,5 + 2\pi)$, or $(-10,5 + \pi)$, or others, where the coordinates do not satisfy the equation $r = 2\theta$.

The polar graph of $r = \sin\theta$ is given in Figure 6-2. Can you now interpret the graph to show that r is a periodic bounded function of θ? We may note that the related polar equation for this graph is $r = -\sin(\theta + \pi) = \sin\theta$, hence is identical with the original polar equation.

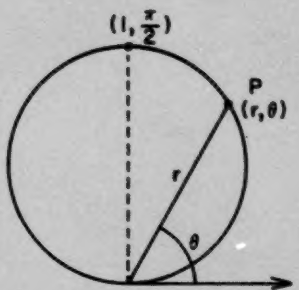


Figure 6-2

Both Figure 6-1 and 6-2, which are graphic representations of $y = \sin x$ exhibit a geometric property called symmetry. The algebraic counterpart of this property will be discussed in detail after the following exercises.

Exercises 6-2(a)

1. Give bounds for the graphs of the following equations.

(a) $y = 2 \sin x$	(f) $y = 0.6 \sin x + 0.8 \cos x$
(b) $y = \sin 2x$	(g) $y = 2 \sin x + 3 \cos x$
(c) $y = 2 - \sin 2x$	(h) $y = a \sin x + b \cos x$
(d) $y = \frac{1}{2} \sin 2x$	(i) $y = \sin^2 x$
(e) $y = 4 + 2 \sin (3x + \frac{\pi}{2})$	(j) $y = \sin^2 x - \cos^2 x$

2. Express in terms of a, b, c, d the bounds and the period of the graph of $y = a + b \sin (cx + d)$.

6-2(b) Symmetry

The graph in Figure 6-1 is symmetric with respect to the origin (and many other points), and to the line $x = \frac{\pi}{2}$ (and many other lines). The graph in Figure 6-2 is symmetric with respect to the point $(\frac{1}{2}, \frac{\pi}{2})$, and to the line $\theta = \frac{\pi}{2}$ (and many other lines). We shall concern ourselves with only the types of symmetry you have already met in earlier courses. We give their definitions here for the sake of completeness.

Point Symmetry. Given a set of points S, and a fixed point M. S is symmetric with respect to M if, for each point P of S there is a corresponding point P' of S such that M is the midpoint of $\overline{PP'}$. (The point P' is called the point-symmetric image of P with respect to M, or, when the context makes the reference clear, the image of P with respect to M.)

Line Symmetry. Given a set of points S, and a fixed line L. S is symmetric with respect to L if, for each point P of S there is a corresponding point P' of S such that L is the perpendicular bisector of $\overline{PP'}$. L is sometimes called an *axis of symmetry* of the set S, which may have more than one such axis. We sometimes borrow terminology from the applications, and call L an *axis of reflection*; in that case we may also call P' the *reflected image* of P with respect to L, or simply, the *reflection* of P in L.

In rectangular coordinates we readily establish an algebraic test for symmetry with respect to the origin. The point $P_1 = (x_1,y_1)$ has the image $P'_1 = (-x_1,-y_1)$ with respect to the origin. If P_1 is on the graph of $f(x,y) = 0$ then $f(x_1,y_1) = 0$. If the graph is symmetric with respect to the origin, for each point $P_1 = (x_1,y_1)$ on it, the graph must also contain the point $P'_1 = (-x_1,-y_1)$. That is, whenever $f(x_1,y_1) = 0$ we must also have $f(-x_1,-y_1) = 0$. This yields our test:

> The graph of an equation in rectangular coordinates is symmetric with respect to the origin if an equivalent equation is obtained by replacing (x,y) by $(-x,-y)$.

We may now test the equation $y = \sin x$, which may be written $y - \sin x = 0$. If we designate the left member as $f(x,y)$, we have: $f(-x,-y) = -y - \sin(-x)$, or $-y + \sin x$, or $-(y - \sin x)$, or $-f(x,y)$. This is clearly equal to zero whenever $f(x,y)$ is equal to zero; therefore, the graph is symmetric with respect to the origin.

As a second example we may test the equation $y = x^3$ whose graph is called a cubic parabola. If we write this equation as $y - x^3 = 0$ and call the left member $f(x,y)$, then we find $f(-x,-y) = (-y) - (-x)^3 = -y + x^3$ $= -(y - x^3) = -f(x,y)$. Clearly this is zero whenever $f(x,y) = 0$, thus our test for symmetry is satisfied, and the graph is symmetric with respect to the origin.

The test for symmetry with respect to any point $M = (h,k)$ other than the origin, is not at all difficult, but will not be presented here. If a curve has such symmetry, we can usually find a simpler analytic representation for it if we use the center of symmetry as a new origin.

In rectangular coordinates we can find a simple algebraic test for symmetry with respect to the axes. The point $P = (x,y)$ has the image $P' = (-x,y)$ with respect to the y-axis, and $P'' = (x,-y)$ with respect to the x-axis.

These relations lead to our test. If the graph is symmetric with respect to the y-axis, then, for each point $P_1 = (x_1,y_1)$ on the graph there must be a point $P_1' = (-x_1,y_1)$ also on the graph; that is, if $f(x_1,y_1) = 0$, so also must $f(-x_1,y_1) = 0$. This means that the equations $f(x,y) = 0$, and $f(-x,y) = 0$ must be equivalent equations. We show that the graph of $y = \sin x$ in rectangular coorinates does not have this type of symmetry. This equation can be written as $y - \sin x = 0$, or $f(x,y) = 0$. Then $f(-x,y)$ is $y - \sin(-x)$ or $y + \sin x$, which clearly need not equal zero when $f(x,y) = y - \sin x$ does.

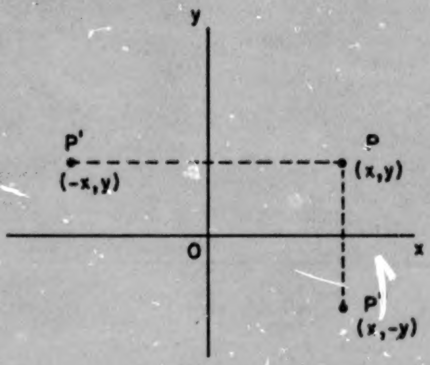


Figure 6-3

The test for symmetry with respect to the x-axis is analogous and we summarize these two tests:

The graph of an equation in rectangular coordinates is symmetric with respect to the

(a) x-axis, if an equivalent equation is obtained by replacing (x,y) by $(x,-y)$;

(b) y-axis, if an equivalent equation is obtained by replacing (x,y) by $(-x,y)$.

It is quite possible for a graph to be symmetric with respect to both axes. The graph of $x^2 + 4y^2 = 36$ is an ellipse and it exhibits such double symmetry both algebraically and geometrically.

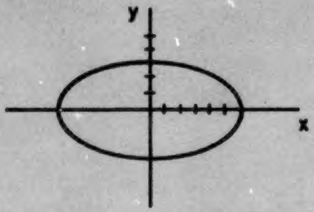


Figure 6-4

If y can be expressed as an explicit function of x, $y = f(x)$, such that $f(x)$ contains only even powers of x then we say that y is an even function of x, and recognize that its graph is symmetric with respect to the y-axis. Some examples of even functions of x are:

$$y = 7x^2 \, , \; y = x^2 + 3x^4 \, , \; y = \sqrt{x^4 - 3x^2} \, , \; y = 2x^2 - \frac{1}{x^6} \, .$$

Note that the equation $x^2 + 4y^2 = 36$ does not define y as a function of x, or x as a function of y. Rather, it yields expressions for y as two (even) functions of x, that is, $y = \frac{1}{2}\sqrt{36 - x^2}$, and $y = -\frac{1}{2}\sqrt{36 - x^2}$. The graphs of these functions are semi-circular arcs each of which is, in fact, symmetric with respect to the x-axis.

Where x and y are related implicitly by an equation $f(x,y) = 0$, we may still use the concepts above. If $f(x,y)$ contains only even powers of x, then $f(x,y) = f(-x,y)$, and the graph of $f(x,y) = 0$ will be symmetric with respect to the y-axis. Thus we may still relate the symmetry of the graph to even functions even when those functions are implicit. Some examples of even implicit functions are:

(a) $x^2y + x^4y^2 = 10$, whose graph is symmetric with respect to the y-axis but not the x-axis;

(b) $x^2y^2 + 3xy^4 + 2x = 0$, whose graph is symmetric with respect to the x-axis but not the y-axis;

(c) $x^2y^4 + 2x^2 + 3y^2 = 4$, whose graph is symmetric with respect to both axes.

Note that the graph of $x^2 + 4y^2 = 36$ is symmetric with respect to the origin also, since $f(x,y) = f(-x,-y)$. Which, if any, of the graphs of a, b, and c, above, are symmetric with respect to the origin?

Symmetry with respect to other lines will not be generally discussed here, but there is a simple test for symmetry with respect to the lines which bisect the angles formed by the axes. These lines are L_1: $y = x$, and L_2: $y = -x$. The reflection of $P = (x,y)$ in L_1 is $P' = (y,x)$, and in L_2 is $P' = (-y,-x)$ as may be seen in the figure.

The corresponding test follows as before and may be stated thus:

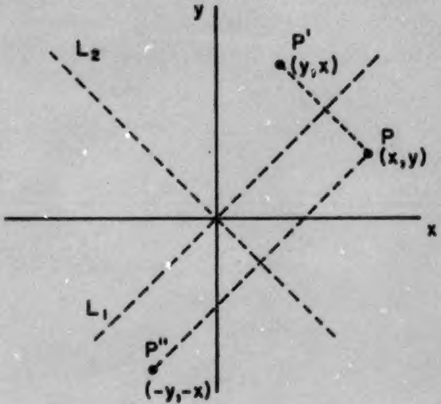


Figure 6-4

The graph of an equation in rectangular coordinates is symmetric with respect to the line,

(a) $y = x$, if an equivalent equation is obtained by replacing (x,y) by (y,x) ;

(b) $y = -x$, if an equivalent equation is obtained by replacing (x,y) by $(-y,-x)$.

Examples:

1. The graphs of the following equations are symmetric with respect to the line $y = x$;

(a) $xy = 6$

(b) $xy = x^3 + y^3$

(c) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

(d) $x + y = 10$

(e) $x^2 + y^2 - 6x - 6y = 12$.

2. The graphs of the following equations are symmetric with respect to the line $y = -x$;

(a) $xy = 6$

(b) $y = x + 3$

(c) $x^2 + y^2 - 6x + 6y = 12$

(d) $x^3 = y^3 + xy$

(e) $y = x^2y^2 + x$.

If a graph has an axis of symmetry parallel to the x-axis or the y-axis it may have a simpler analytic representation if we use new coordinates based on this axis of symmetry. Such transformations of coordinates are considered in detail in Chapter 10. Tests for symmetry with respect to other lines than those mentioned are available, but they are beyond the scope of this book.

These comments on symmetry in rectangular coordinates have their counterparts in polar coordinates. Point symmetry with respect to the pole requires that the graph of $f(r,\theta) = 0$ contain, for each point $P = (r_1,\theta_1)$ the corresponding point $P' = (-r_1,\theta_1)$. This condition will be satisfied if $f(r,\theta)$ is an even function of r . Note that the condition is sufficient to establish such symmetry but it is not necessary. Thus, the graph of $r = 5$ is a circle with radius 5 , and it does have such symmetry, but this equation does not define an even function of r . We will not analyze the general situation, but note that $r = 5$ and $r = -5$ are related polar equations for the same circle. These equations may be written as $r - 5 = 0$ and $r + 5 = 0$, and then combined as in Chapter 5 by multiplying corresponding members to get $r^2 - 25 = 0$. This equation does give an even function of r and its graph, which is the same as that of $r = 5$ and of $r = -5$, is therefore symmetric with respect to the pole.

The point $P = (r,\theta)$ has, as its image with respect to the line containing the polar axis, the point $P' = (r,-\theta)$. We will not treat line symmetry in general, but we note an easy test for symmetry with respect to any line through the pole, say the line $\theta = k$. In this case the points $P = (r,k + \alpha)$ and $P' = (r,k - \alpha)$ are line-symmetric images for any value of α .

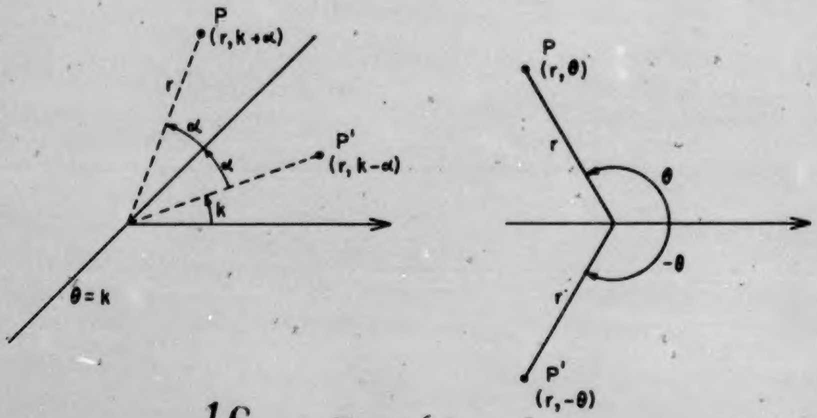


Figure 6-5

We state a test for such symmetry:

The polar graph of an equation is symmetric with respect to the line $\theta = k$ if an equivalent equation is obtained by replacing $(r,k + \alpha)$ by $(r,k - \alpha)$. In particular, the graph will be symmetric with respect to the line along the polar axis if $f(r,\theta) = f(r,-\theta)$.

These should again be recognized as sufficient but not necessary conditions. Since we have infinitely many polar representations of the symmetric points P and P' we could have infinitely many test for such symmetry. The test we have presented is the simplest to apply, and, with the concept of related polar equations, is adequate for the work of this course.

If we go back to an equation from Example 1, $r = \sin \theta$, we may write it $r - \sin \theta = 0$, and call the left member of this equation $f(r,\theta)$. The diagram there suggests that the line $\theta = \frac{\pi}{2}$ is an axis of symmetry and we compare $f(r,\frac{\pi}{2} + \alpha)$ and $f(r,\frac{\pi}{2} - \alpha)$. The first of these becomes $r - \sin(\frac{\pi}{2} + \alpha)$, or $r - \cos \alpha$. The second of these becomes $r - \sin(\frac{\pi}{2} - \alpha)$ or $r - \cos \alpha$. The identity of these expressions established the line symmetry of the graph, as indicated. We may have stated, in corresponding manner, that the point $P = (r,\frac{\pi}{2} + \alpha)$ is on the curve if and only if the corresponding point $P' = (r,\frac{\pi}{2} - \alpha)$ is on the curve. This is, in effect, what we have shown.

Exercises 6-2(b)

1. May a set of points have two centers of symmetry? Discuss your answer, with examples.
2. Give an example of a set of points which has exactly 2 axes of symmetry; exactly 3 ; exactly 4 .
3. Give an example of a set of points which has an infinte number of axes of symmetry.
4. If a graph is symmetric with respect to both axes must it be symmetric with respect to the origin? Illustrate.
5. If a graph is symmetric with respect to the origin must it be symmetric with respect to both axes?

6. Discuss the symmetry of the graphs of each of the equations listed:

(a) $x^2 + y^3 = 16$

(b) $x^3 - y^3 = x + y$

(c) $y = x^2 - 2x^4 + 5x^6$

(d) $x(x^2 + y^2) = y(x^3 + y^3)$

(e) $x^2y + xy^2 = 1$

(f) $(x + y)^2 + 2(x + y) = 1$

(g) $(x + y)^2 + 3(x + y) = 1$

(h) $x^2 + y^3 = y^2 + x^3$

(i) $x^4 + x^2y^2 + y^4 = x + y^2$

(j) $x^n + y^n = 1$

(k) $r^2 = \sin \theta$

(l) $r = \sin^2 \theta$

(m) $r = 2 + \sin(\theta + \pi)$

(n) $r = \frac{6}{1 + \cos \theta}$

(o) $r = \frac{6}{3 - \cos(\theta - \frac{\pi}{2})}$

(p) $r^2 \cos^2 \theta = 10$

(q) $r^2 = \sin 2\theta$

(r) $r = 2 \sin 3\theta$

(s) $r = 3 + 2 \cos(\theta + \frac{\pi}{2})$

(t) $r = a + b \sin \theta$

Challenge Problems

1. (For discussion) By analogy with line symmetry in two dimensions, consider symmetry with respect to a plane in three dimensions. We are familiar with our reflected images in a mirror and accept the fact that there is a "reversal" of some sort. The reflection of my right hand is the "left hand" of my reflected image. Why is this reversal only left-right? Why is there not also a reversal of top-bottom, so that my reflected image would appear to stand on its head?

2. Given the line L: $ax + by + c = 0$ and the point $P_1 = (x_1, y_1)$ not on the line. Find coordinates for $P_2 = (x_2, y_2)$, the symmetric image of P_1, with respect to L.

6-2(c) Extent.

We discussed the equation $x^2 + 4y^2 = 36$ earlier from the point of view of symmetry. We use it now to discuss the extent of a graph. This equation yields two equations which define y as a function of x, $y = \frac{1}{2}\sqrt{36 - x^2}$ and $y = -\frac{1}{2}\sqrt{36 - x^2}$. We see that if we take values of

$|x|$ large enough we shall have in both cases corresponding values of y which are imaginary. Since our graphs consider only real values of x and y we now inquire about possible values of x which will lead to real values of y, and vice versa. In these cases we must have $-6 \leq x \leq 6$, or $|x| \leq 6$. For these restricted values of x the corresponding values of y range from -3 to 3. The geometric versions of these restrictions can be applied to the graphs of both functions of x defined above, but it is more useful to consider the union of these graphs; that is, the graph of the original equation

$$x^2 + 4y^2 = 36 .$$

From the discussion above we see that the points of the graph all lie in a rectangular region 12 units wide and 6 units high, centered at the origin. If, in general, we can express y as a function of x, and there are such restrictions on values of x as will yield only real values for y, we say that the domain of the function is bounded. Thus, all points of the graph of the function $y = \frac{1}{2}\sqrt{36 - x^2}$ are confined to a strip bounded by two vertical lines, $x = \pm 6$, as indicated in Figure 6-6. If, in general, the possible real values of y are similarly restricted, we say that the range of the function is bounded. Thus, all points of the graph of $y = \frac{1}{2}\sqrt{36 - x^2}$ are confined to a strip bounded by two horizontal lines, $y = \pm 3$, as indicated in Figure 6-6. If both the domain and range of a function are bounded, we say that the function is bounded, in which case its graph is confined to the intersection of a vertical and horizontal strip, and is therefore confined to a rectangular region. These terms are usually applied to equations and their graphs even when the functions are only defined implicitly. Thus, when we say that the graph of $x^2 + 4y^2 = 36$ is bounded, we indicate that it is contained in a rectangle, as mentioned earlier.

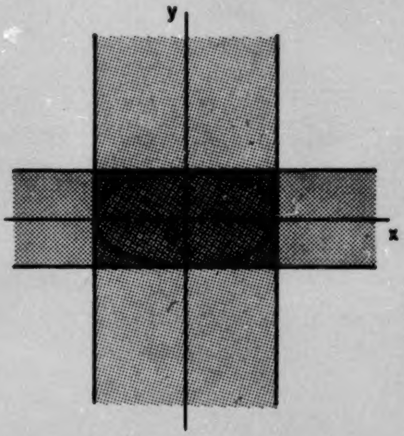


Figure 6-6

If the equation were $x^2 - 4y^2 = 36$, we would obtain

$$y = f(x) = \pm \frac{1}{2} \sqrt{x^2 - 36} .$$

We now note that we must take values of $|x|$ large enough to make the radicand non-negative; that is, $|x| \geq 6$, which will be true if either $x \geq 6$ or $x \leq -6$. Geometrically, this means that y is defined only for points on the edges or outside the vertical strip bounded by the lines which are the graphs of $x = 6$ and $x = -6$. With these restrictions on x we may now have any value of y . The original equation yields two equations which define x as a function of y , $x = \sqrt{36 + 4y^2}$ and $x = -\sqrt{36 + 4y^2}$ and we see that x in both cases is defined for all values of y . It is not customary, in this case, to speak of $y = \frac{1}{2} \sqrt{x^2 - 36}$ as a bounded function, but merely to say that the domain of x excludes certain values.

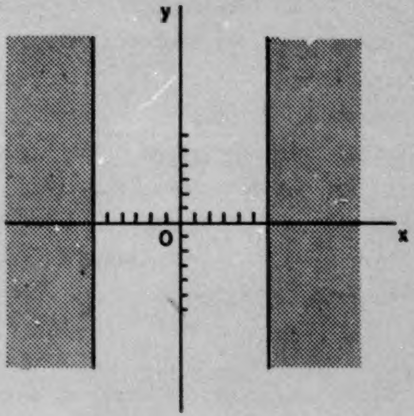


Figure 6-7

Another concept emerges when we consider $y = \frac{1}{x}$. The domain of x is also restricted here since x cannot equal zero. With this exception, y is defined for all values of x . Geometrically, points of the graph are available except at the places where the abscissa is zero, therefore this graph does not touch or cross the y-axis. If we write the equation $x = \frac{1}{y}$, we see that the graph does not touch or cross the x-axis. Also, from the fact that $xy = 1$, we must have x and y either both positive or both negative, which means, geometrically, that we are confined to the first and third quadrants exclusively. From the equation $xy = 1$ we see also that as we take points of the graph nearer the x-axis we must take them farther from the y-axis, and vice-versa. A line, such as the x-axis in this case, to which points of the graph approach more and more closely, but which contains no point of the graph, is called an asymptote of the graph. The graph of $y = \frac{1}{x}$ has two asymptotes; namely, the x-axis and the y-axis. Our examples will illustrate the treatment of asymptotes in several situations, but we make a general observation. If our analytic representation can be written as

$$y = \frac{f(x)}{g(x)} ,$$

where $g(x)$ may equal zero for some value of x, say $x = a$ then, for this value of x, y is not defined. Also, if, $f(b) \neq 0$ then, in general, as we take values of x closer to b the corresponding values of y become greater in absolute value. Geometrically this usually means that as we take points closer to the line $x = b$ they must be farther from the x-axis. Thus, the line $x = b$ is a vertical asymptote. If $g(x) = 0$ has roots b_1, b_2, ..., and these are not roots of $f(x) = 0$, there will, in general, be vertical asymptotes, $x = b_1$, $x = b_2$, ... There is no difficulty in revising these comments to apply to horizontal asymptotes: If we can write $x = \frac{h(y)}{k(y)}$; and $k(y) = 0$ has roots c_1, c_2, ..., and these are not roots of $h(y) = 0$, then, in general, there will be horizontal asymptotes, $y = c_1$, $y = c_2$, ...

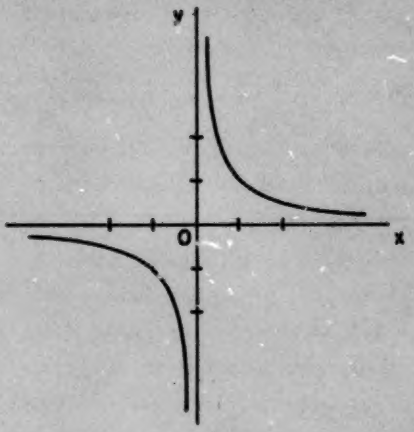


Figure 6-8

Example: Discuss and sketch the graph of

$$y = \frac{x}{x^2 + 2x - 3} .$$

Solution: The equation can be written as $y = \frac{x}{(x + 3)(x - 1)}$; hence, from the discussion above, the curve has as vertical asymptotes the lines $x = -3$ and $x = 1$. y is not defined for these values of x, but y is defined for all other values of x. If $x > 1$ and increasing then y is positive, and decreasing. For large values of x the values of $x + 3$ and $x - 1$ are relatively close to values of x , and y is relatively close to $\frac{1}{x}$, which is positive; therefore, the corresponding points of the curve are close to the x-axis. If $0 < x < 1$ the numerator is positive and the denominator negative; therefore, y is negative. The curve still approaches the line $x = 1$ as an asymptote, but from the other side. If $-3 < x < 0$ the numerator and denominator are both negative, therefore y is positive. As before, the curve approaches the line $x = -3$ as an asymptote. If $x < -3$ then the numerator is negative, the denominator positive, and y negative. The curve again approaches the line $x = -3$ as an asymptote, but from the left side.

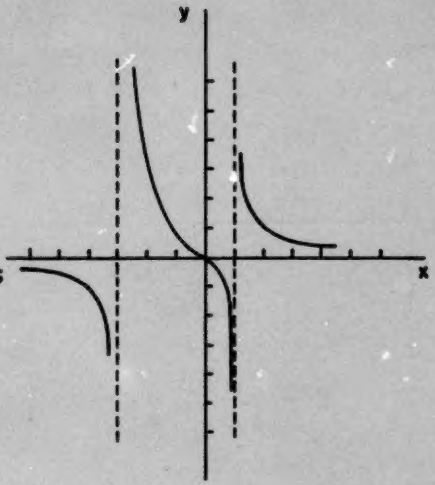


Figure 6-9

For negative values of x with large absolute value the values of $x + 3$ and $x - 1$ are relatively close to x , and the corresponding value of y is relatively close to $\frac{1}{x}$, which is now negative. That is, as we take points of the graph farther to the left, they must be closer to the x-axis, from below. The graph, pictured in Figure 6-9, shows that algebraic and geometric relationships we have discussed.

A discussion of the appearance of a graph for large values of $|x|$ or $|y|$, whether we take x and y positive or negative, is part of the discussion of the extent of the graph, and is sometimes referred to as a discussion of the behaviour of the graph for extreme values of the variables.

The concept of excluded values because of a zero denominator has one further application. Consider

$$y = x + 2 \text{ , and } y = \frac{x^2 - 4}{x - 2} .$$

It would not be correct to write the second equation as

$$y = \frac{(x + 2)(x - 2)}{(x - 2)}$$

and then remove the common factor

$$y = \frac{(x + 2)\cancel{(x - 2)}}{\cancel{(x - 2)}}$$

to arrive at the first equation

$$y = x + 2 .$$

As a matter of fact, the two equations and their graphs are different in a small but significant way. In the first equation, y is defined for all x ; in the second equation y is defined for all x , except $x = 2$. Geometrically, the graph of the first equation is a line; the graph of the second equation is a line except for a missing point at the place where $x = 2$, that is, it is an interrupted line. (Could you interrupt this line at the place where $x = 1$, also?)

The discussion of these excluded points, lines, or regions is useful in describing the extent of the graph. It's all very well to know where the graph does not go, but we are still concerned with the points through which it does go, that is, with drawing the graph. The most straightforward way of drawing the graph of an equation is to plot a number of points on it and draw a curve through them. If the equation has the form $y = f(x)$ you can make a table showing the value of y corresponding to each of a number of values of x . You have done this many times in the past, and there is no need to go into detail again here. However, it is worth reminding you that you should think about how many values of x to use, and which ones, and how to join the corresponding points.

As in an election poll, we take enough samples, with special attention to certain critical spots, until we have some reasonably clear idea of how the whole picture will look. There will always be some disagreement about how many are "enough", and what is "reasonably clear". Our sampling can start at some easily available points. On our grid we can most easily find the places where the graph crosses the axes. Since the x-axis, for example, has the equation $y = 0$, we may solve simultaneously: $y = 0$, $y = f(x)$; that is, we may find the roots of the equation $f(x) = 0$, in order to find the abscissas of these crossing points. If $f(x) = 0$ has roots a_1 , a_2 , ... , then these numbers are the x-intercepts of the graph, which goes through the points $(a_1,0)$, $(a_2,0)$, ... These points are easily plotted on the grid, as are the points of intersection of the graph with the y-axis. But, no matter how many points you plot, there always remains the question of how the curve behaves elsewhere. It is to cast further light on this question that you should investigate, before any extensive computation, the properties of the curve and its analytic representation in the manner we have just illustrated. We summarize this type of investigation in mnemonic form: "Check the SEPIA first." (Symmetry, Extent, Periodicity, Intercepts, Asymptotes.)

The curves and equations with which we deal in this course are reasonably well behaved, and the points of the graph are usually smoothly connected, with certain notable exceptions. We have already dealt with graphs of inequalities in Chapter 5 , and will not deal with them at great length here, but will consider them in the examples whenever there is any matter of special interest.

A curve usually separates the plane locally into two regions (above and below, inside and outside, ...). In many cases in this text the points in these two regions are precisely those whose coordinates satisfy one or the other of the inequalities we obtain from the original equation. Thus the graph of $x^2 + y^2 = 25$ is a circle of radius 5 , centered at the origin. The graph of $x^2 + y^2 < 25$ is the interior of that circle, and the graph of $x^2 + y^2 > 25$ is the exterior.

We have used rectangular coordinates in this general discussion, but much of it can be adapted to polar coordinates, though the graphs will not have the same geometric properties. In polar coordinates the graphs of inequalities are sometimes unexpected. Thus the graph of $r = 5$ is a circle, the graph of $r > 5$ is the region outside that circle, but the graph of $r < 5$ is the entire plane. The graph of $r = \frac{1}{\theta}$ is only a remote cousin to the graph of $y = \frac{1}{x}$. The rectangular graph (a hyperbola) has a vertical asymptote, the line $x = 0$, and this is a geometric consequence of the fact that y is not defined for $x = 0$. From the equation $r = \frac{1}{\theta}$, we see that r is not defined for $\theta = 0$; nevertheless the line $\theta = 0$ contains the point $P = (-\frac{1}{\pi}, 0)$. This point has infinitely many other polar representations, including particularly $P = (\frac{1}{\pi}, \pi)$, and since these coordinates satisfy the equation $r = \frac{1}{\theta}$, we must allow P on the graph of $\theta = 0$. There are, as a matter of fact, infinitely many other points for which we can find some pair of polar coordinates that satisfy $r = \frac{1}{\theta}$, and which lie on the line $\theta = 0$. Therefore this line is not an asymptote for the graph if $x = \frac{1}{\theta}$.

The graph of $r = \frac{1}{\theta}$ does, nevertheless, have a true asymptote, the line corresponding to $r = \frac{1}{\sin\theta}$, but the discussion of this must consider the value of $\frac{\sin\theta}{\theta}$ as θ gets closer to 0 , and this discussion is beyond the scope of this book.

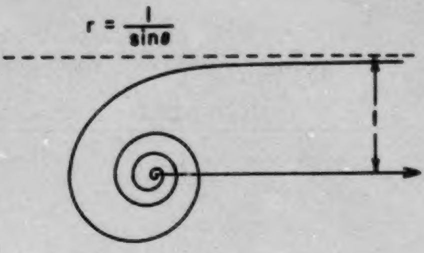


Figure 6-10

We will, in the examples and text that follow, use polar representation or any other that seems appropriate to the problem and our purposes, and carry the discussion to the level and detail that seem fitting. Our examples will illustrate the general principles above, and some ideas of less general application, but the student is urged to extend his own experience by doing as many of the exercises as he can. One suggestion we have found valuable: an equation and its graph should be considered in a dynamic, rather than a static way. If we have $y = f(x)$, what happens to y when x increases a little, when x approaches 0 , when x gets very large? If we have a point $P_0 = (x_0,y_0)$ of the graph, how does the curve look, near that point? Think of the point as moving along the curve, and our analysis as a moving picture of the point rather than a snapshot of the entire curve.

6-3. Conditions and Graphs (Rectangular Coordinates)

In this section we shall discuss a number of examples in detail. This discussion will bring together and apply a number of topics you first studied separately. We shall illustrate also some useful approaches that may be new to you.

Example 1. Discuss and sketch the graph of $y = x + \frac{1}{x}$.

Solution. There is no symmetry with respect to either axis, since we do not get equivalent equations by replacing x by $-x$; or y by $-y$. There is symmetry with respect to the origin, because we do get an equivalent equation by replacing x by $-x$ and y by $-y$. There is a vertical asymptote, the y axis, whose equation is $x = 0$. For large $|x|$ and x either positive or negative, y and x become relatively equal, since $\frac{1}{x}$ becomes relatively small. Geometrically this means that the graph approaches the line $y = x$ asymptotically, from above, on the right, and from below, on the left.

We shall graph this equation in a way which may be new to you, by addition of ordinates. You can draw fairly accurate graphs of $y = x$ and $y = \frac{1}{x}$ with almost no effort. Do so, with respect to the same axes. Then, for each of a number of different values of x , add the y-coordinates of the points on the two curves with that x-coordinate. The result is the y-coordinate of the corresponding point on the graph of $y = x + \frac{1}{x}$. The addition can be done using marks on the edge of a piece of paper, but you must pay attention to the algebraic signs. The sketch below illustrates the process.

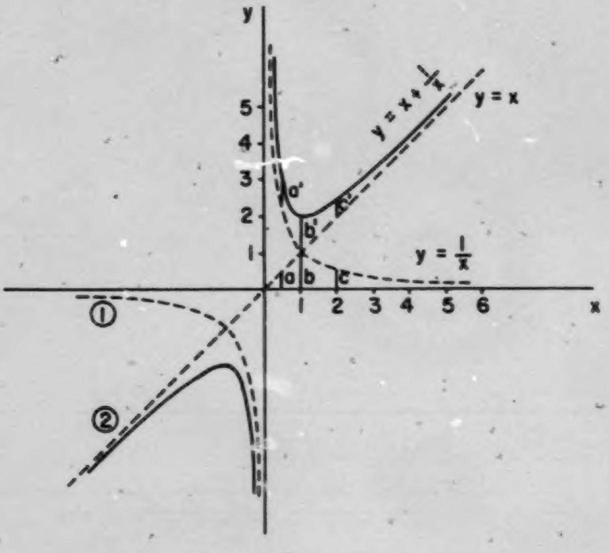


Figure 6-11

We suggest this sequence of steps:

1. Draw the familiar curves ① and ②.
2. At several points along the x-axis erect perpendiculars to meet the two curves. In Figure 6-11 the ordinate segments, a , b , c , were found this way at $x = \frac{1}{2}$, $x = 1$, $x = 2$. (We shall refer to these ordinate segments simply as the ordinates.)
3. Add the corresponding ordinates for the two curves with due regard to sign. In Figure 6-11, a , the ordinate at $x = \frac{1}{2}$ is raised to a' above the hyperbola; b is raised to b' above the hyperbola; c is raised to c' above the line; and so on.
4. Connect the new points thus found, to get the new curve.

Example 2(a) Sketch the graph of $y = x^2 + 2$.

Example 2(b) Sketch the graph of $y = \sin x - 3$.

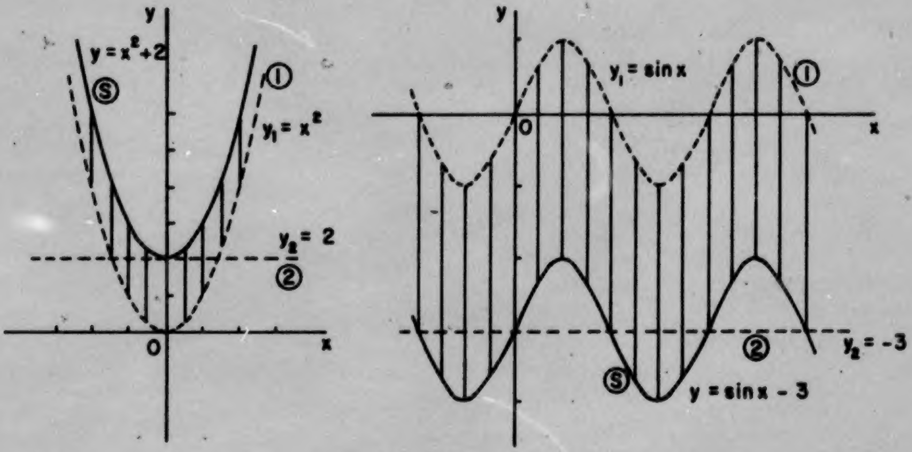


Figure 6-12

Solution 2(a). Draw the familiar graphs of $y_1 = x^2$, indicated by ① in the figure and of $y_2 = 2$, indicated by ② in the figure. Then "raise" every point of ① 2 units, as indicated by the dashed lines, to get the graph Ⓢ of $y = y_1 + y_2 = x^2 + 2$.

2(b) The solution should be clear from the figure and is left to the student.

The process of graphing by subtraction of ordinates is related to the process of graphing $y = -f(x)$ from the graph of $y = f(x)$. The discussion of symmetry in the previous section indicates immediately that these two graphs are symmetric images of each other with respect to the x-axis. That is, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$, with respect to the x-axis.

Example 3(a). Sketch the graph of $y = -x^2$.

Example 3(b). Sketch the graph of $y = -\cos x$.

Solution: (Refer to Figure 6-13).

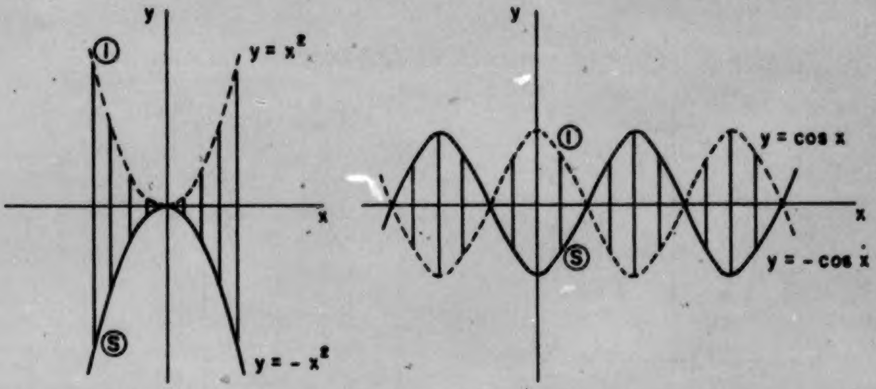


Figure 6-13

3(a) Construct the familiar graph ① of $y = x^2$; then extend the ordinate of each point of ① down its own length through the x-axis to get the reflected points, which we connect to obtain the solution, Ⓢ.

3(b) The solution, indicated in Figure 6-13, is left to the class.

We may now sketch graphs by subtracting ordinates, since, if y $y = f(x) - g(x)$, then $y = f(x) + (-g(x))$.

__Example__ __4(a)__. Sketch the graph of $y = 3 - x^2$.

__Example__ __4(b)__. Sketch the graph of $y = 1 - \sin x$.

__Solution__ __4(a)__. (Refer to Figure 6-14).

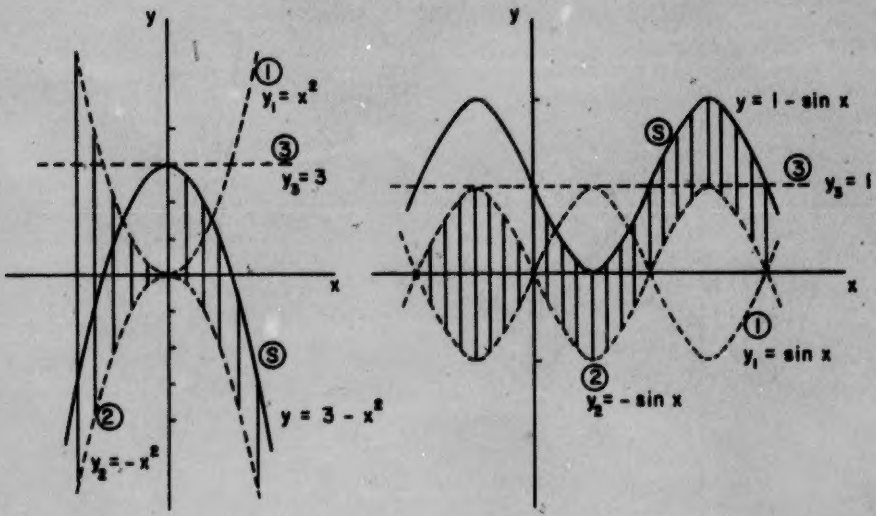


Figure 6-14

We suggest these steps:

(1) Draw the familiar graphs ①: $y_1 = x^2$, and ③ $y_3 = 3$.

(2) Reflect ① with respect to the x-axis to get ②: $y_2 = -x^2$.

(3) Add the ordinates for ② and ③ to get Ⓢ: $y = 3 - x^2$.
This last step is equivalent to adding 3 units to each ordinate of ②, as indicated on the graph.

We may extend these graphical methods to the multiplication of ordinates. We have already done this in some cases but not with this terminology. The graph of $y = 2 \sin x$ illustrates a simple application of this method. We compare this graph with the graph of $y_1 = \sin x$ and recognize that when $y_1 = 0$ then $y = 0$; when $y_1 > 0$ then $y > 0$; and when $y_1 < 0$, then $y < 0$. We just draw the graph of $y_1 = \sin x$, and __double__ the ordinates to find corresponding ordinates for $y = 2 \sin x$. It is as if the graph were stretched, vertically, away from the x-axis.

Example 5(a). Sketch the graph of $y = 2 \sin x$.

Example 5(b). Sketch the graph of $y = 2x^2 - 8$.

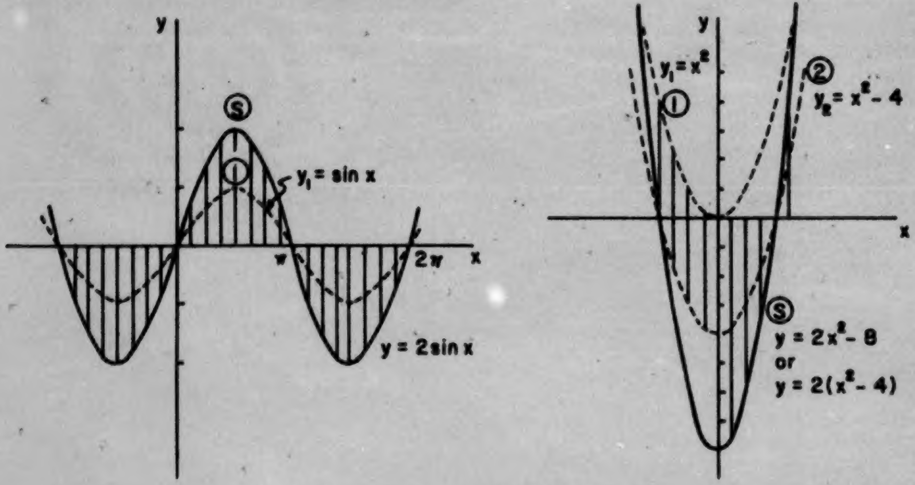


Figure 6-15

Solution 5(a). We sketch the familiar graph, ① : $y_1 = \sin x$, then double each ordinate of ① to get the graph, Ⓢ : $y = 2 \sin x$. Note that for $0 < x < \pi$ we have $0 < y_1 < 1$, therefore $0 < 2y_1 < 2$. Thus Ⓢ is bounded between 2 and -2 . If, more generally, $y = a \sin x$, then y is bounded between $|a|$ and $-|a|$. In this case $|a|$ is called the amplitude of this sine curve. It is the measure of the maximum departure of points of the curve from the x-axis, and has important physical applications.

Solution 5(b). We have illustrated the sequence of graphs: ① : $y_1 = x^2$; ② : $y_2 = x^2 - 4$; ③ : $y = 2(x^2 - 4)$. We could have found the same graph with the sequence ① : $y_1 = x^2$; ② : $y_3 = 2x^2$;

③: $y = 2x^2 - 4$. We leave the details to the student.

We may in general relate the graph of $y = bf(x)$ to that of $y_1 = f(x)$ if b is a constant. Both graphs cross the x-axis at the same points. If $b > 0$ then both graphs are above or below the x-axis together. If $b < 0$ then the graphs of $y = -bf(x)$ and $y_1 = f(x)$ are together above or below the x-axis. In this latter case we graph $y_2 = |b|\, f(x)$, then reflect this graph in the x-axis to get the graph of $y = bf(x)$.

Example 6(a). Sketch the graph of $y = -2x^2$.

Example 6(b). Sketch the graph of $-3 \sin x$.

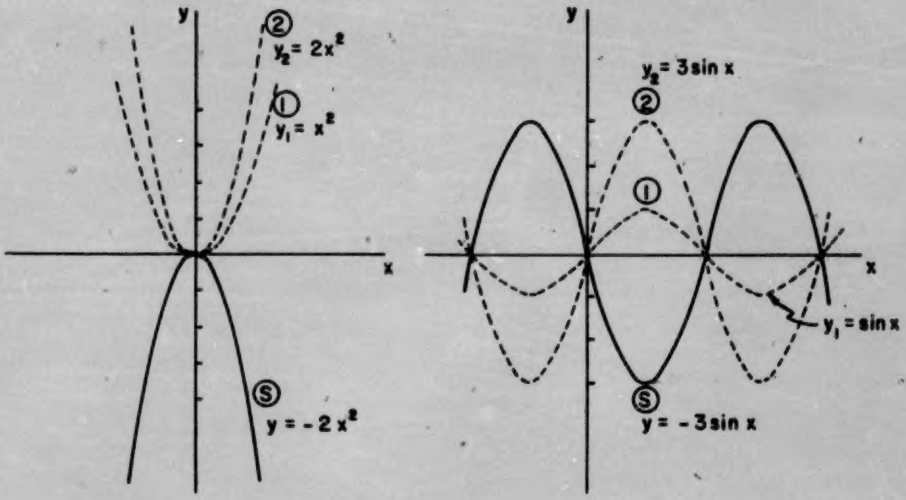


Figure 6-16

Solution 6(a). Sketch the familiar curve ①: $y_1 = x^2$. Double the ordinates, which in this case are all non-negative, to get ②: $y_2 = 2x^2$. Finally reflect ② in the x-axis to get Ⓢ: $y = -2x^2$.

Solution 6(b). We leave the solution to the student. Note that in Example 6(a) we could have used the sequence $y_1 = x^2$; $y_3 = -x^2$, $y = -2x^2$. That is we could have reflected, then stretched to get the final curve, in both 6(a) and 6(b) . We leave these details to the student.

Our final cases concern multiplication of ordinates with variable factors. These are the most difficult, the most interesting, and the most useful of the applications of these methods of graphing by combinations of ordinates.

Example 7. Sketch the graph of $y = x^2 - x$.

Solution.

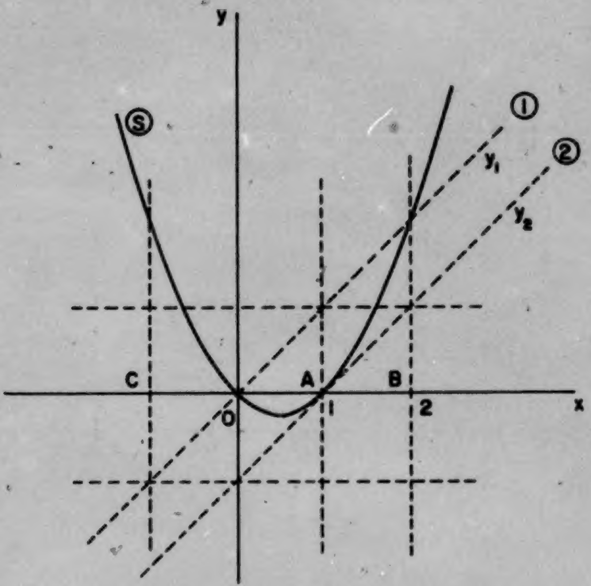


Figure 6-17

We could sketch the graph by subtraction of ordinates but we choose to illustrate the method of graphing by multiplication of ordinates. Thus $y = x(x - 1)$, and we draw the graphs ①: $y_1 = x$, and ②: $y_2 = x - 1$; two parallel lines. When $x < 0$ then y_1 and y_2 are both negative and their product, y , is positive. If $x < 0$ and decreasing then y is positive and increasing, and corresponding points of S are in the third quadrant.

Since $y = y_1 y_2$, clearly y must equal zero when either y_1 or y_2 equals zero, thus the graph S intersects the x-axis at A and B . Between O and A we have $0 < x < 1$, with ① above and ② below the x-axis. In this interval $y_1 > 0$, $y_2 < 0$ and therefore $y < 0$ and the graph is below the x-axis. Between A and B we have $1 < x < 2$ and both y_1 and y_2 positive, therefore $y > 0$. The graph indicates that since ① and ② are above the x-axis then Ⓢ must be also. However in that interval $0 < y_2 < 1$ therefore y_2y_1 is a proper fractional part of y_1 , thus $y = y_2y_1 < y_1$, therefore Ⓢ is above ② but below ①.

As x increases beyond B we have $x > 1$, y_1 and y_2 positive and increasing, and y increasing even more rapidly, thus Ⓢ is above both ① and ②.

We have taken this time to discuss the graph of what is, after all, only a parabola, because the analysis and method will help in more difficult and unfamiliar situations.

Example 8. Sketch the graph of $y = .1x \sin x$.

Solution. We are familiar with the graphs of $y_1 = .1x$, and $y_2 = \sin x$. Since $\sin x$ is a bounded periodic function of x we have $|y_2| \leq 1$ and $|y_1| \leq |.1x|$. The graph of this last condition is the pair of lines ① and ② in Figure 6-18.

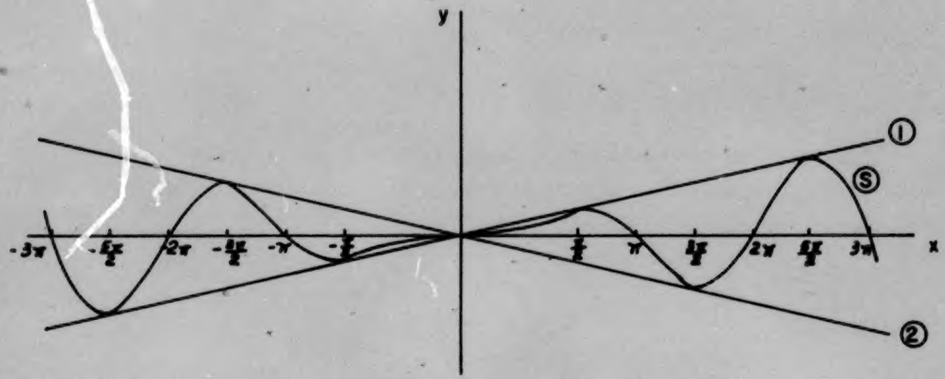


Figure 6-18

We have compressed the scale along the x-axis for the purpose of getting enough of the graph on the page to illustrate the discussion.

When $x > 0$, all points of the graph lie within, or on the boundary of the angular region formed by the right half-lines of ① and ②. Since $y = y_1 y_2$, then y will equal zero when either y_1 or y_2 equals zero. y_1 is zero only at the origin, but y_2 is zero at integral multiples of π. Also, when $y_2 = 1$ we have $y = .1x$ and when $y_2 = -1$ we have $y = -.1x$, which means that the graph Ⓢ will touch alternately the lines 1 and 2 at points where $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

We leave the rest of the discussion of this graph to the student but mention an important application.

If we consider how the graph of $y_2 = \sin x$ is changed by the variable factor $y_1 = .1x$, we may think of the amplitude of y_2, as changed by this variable factor. In this example we may say that the amplitude of $\sin x$ is increasing linearly. If we had $y_3 = f(x) \sin x$ then we also have a sine wave whose amplitude is being changed or constrained by the variable factor $f(x)$. The graph of y_3 would be constrained by the symmetric curves: $y = f(x)$ and $y = -f(x)$ and would oscillate between them, touching them alternately when $x = \pi, 3\pi, 5\pi, \ldots$, as before.

This systematic changing of the amplitude is called amplitude modulation and is the basis for AM radio reception. A typical equation here would be $y = \sin 1000\pi t \sin 1000000\pi t$.

This graph would show a rapidly oscillating curve (the carrier or radio frequency, or RF wave) modulated by a less rapidly oscillating curve (the signal, or audio frequency, or AF wave).

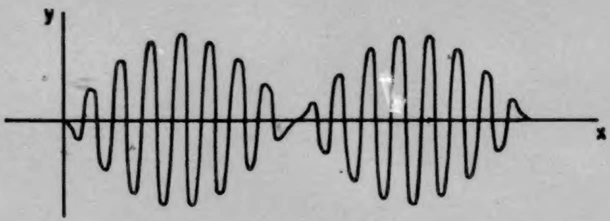


Figure 6-19
This sketch, not to scale, illustrates the idea.

The methods just discussed, for relating graphs of equations to graphs of more familiar equations by combining ordinates are called by some authors, composition of ordinates. We apply similar techniques in polar graphs in some examples later.

We consider now some further examples of graphs of equations in rectangular coordinates.

Example 9. $4x^2 - 9y^2 + 8x + 36y + 4 = 0$. From this equation it is not obvious whether the curve is symmetric with respect to any point or line, or whether it has any asymptotes. Nor can we easily see what parts of the plane it does or does not enter. We can find as many points on it as we have the patience for, since picking a value for x gives us a quadratic equation for y .

The sensible approach, however, is to use a trick you learned in algebra: complete the square in x and y . We get

$$4(x^2 + 2x + 1) - 9(y^2 - 4y + 4) = -4 + 4 - 36$$

or

$$\frac{(y - 2)^2}{4} - \frac{(x + 1)^2}{9} = 1 .$$

These numerators are related to distances from the lines $y = 2$ and $x = -1$, and we might expect a considerable simplification in the discussion of this graph if we had new coordinates based on these lines as axes. Such transformations are carried out more generally in Chapter 10, but we show the details here in order to continue with our discussion of the graph. If we let $u = x + 1$ and $v = y - 2$ the equation becomes

(1)
$$\frac{v^2}{4} - \frac{u^2}{9} = 1 .$$

This equation is considerably easier to handle, and is recognized as an equation of a hyperbola. You know something about hyperbolas, but we continue with our general approach so that after you have seen it work in familiar situations you may be able to use it in unfamiliar ones.

The graph is symmetric with respect to both new axes, and hence with respect to the origin. If we solve (1) for v in terms of u we get $v = \pm \frac{2}{3} \sqrt{u^2 + 9}$. This makes it clear that for a large, positive value of u , the two values of v are one large and positive, the other large and negative. (1) also shows that if (u,v) is any point on the graph, then $|v| \geq 2$. For

$\frac{u^2}{9} \geq 0$, and since $\frac{v^2}{4} - \frac{u^2}{9} = 1$, $\frac{v^2}{4} \geq 1$. Thus no point of the graph lies above $v = -2$ and below $v = 2$.

Now let us consider the part of the curve which lies in the first quadrant. For this we can use the equation

$$v = \frac{2}{3}\sqrt{u^2 + 9}$$

where $u \geq 0$. It seems almost obvious that when u is large, v is very nearly equal to $\frac{2}{3}u$. We can confirm this guess quite simply. Clearly $v > \frac{2}{3}u$, so let us consider $v - \frac{2}{3}u$, in the hope that we can prove it approaches 0 as u grows very large.

$$\begin{aligned} v - \frac{2}{3}u &= \frac{2}{3}\sqrt{u^2 + 9} - \frac{2}{3}u \\ &= \frac{2}{3}(\sqrt{u^2 + 9} - u) \\ &= \frac{2}{3}\frac{(\sqrt{u^2 + 9} + u)(\sqrt{u^2 + 9} - u)}{\sqrt{u^2 + 9} + u} \\ &= \frac{2}{3}\,\frac{u^2 + 9 - u^2}{\sqrt{u^2 + 9} + u} \\ &= \frac{6}{\sqrt{u^2 + 9} + u} \ . \end{aligned}$$

By taking large enough values of u we can make $v - \frac{2}{3}u$ as near to zero as we like. Thus we have shown that in the first quadrant, the graph lies above the line $v = \frac{2}{3}u$ but arbitrarily close to it for large enough u . In other words, $v = \frac{2}{3}u$ is an asymptote of the curve. By similar arguments we can show that $v = \frac{2}{3}u$ is also asymptotic to the part of the curve in the third quadrant, and that $v = -\frac{2}{3}u$ is asymptotic to the parts of the curve in the second and fourth quadrants.

The results above have been stated in terms of the new coordinates. They can easily be restated in terms of the old. For example, the asymptotes are the lines $y - 2 = \pm\frac{2}{3}(x + 1)$.

Finally we consider the intercepts. Setting $u = 0$ in (1) we get $\frac{v^2}{4} = 1$, so the v-intercepts are 2 and -2 . Setting $v = 0$ we get $-\frac{u^2}{9} = 1$, which has no solution. Hence the curve does not intersect the u-axis. The x- and y-intercepts can be found by the same sort of procedure, but since we are chiefly interested in sketching the curve, let's not bother with them.

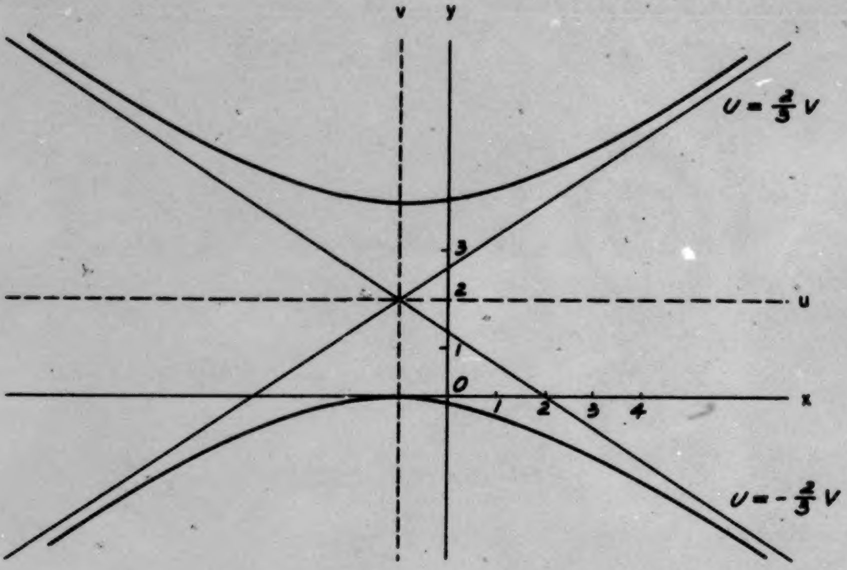


Figure 6-20

The hyperbola is sketched above. Notice that we can draw a fairly accurate graph without finding the coordinates of any points but the vertices. (What are the vertices of a hyperbola?)

When you first studied the hyperbola you learned that the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 .$$

This is an illustration of a principle which is sometimes useful in sketching loci. It can be expressed loosely in the following way. If $f(x,y) = g(x,y) \cdot h(x,y)$, the graph of $f(x,y) = 0$ is the union of the graphs of $g(x,y) = 0$ and $h(x,y) = 0$. Thus since

$$x^2 - y^2 - x + 5y - 6 = (x - y + 2)(x + y - 3)$$

the graph of

$$x^2 - y^2 - x + 5y - 6 = 0$$

is the pair of the lines which are the graphs of

$$x - y + 2 = 0$$

and

$$x + y - 3 = 0 .$$

Before trying to prove the principle we had better find out more accurately what it says. Let's "factor" $x + y$:

$$x + y = (x^2 - y^2) \cdot \frac{1}{x - y} .$$

Unfortunately, the graph of

$$x + y = 0$$

is a line, the graph of

$$x^2 - y^2 = 0$$

is two lines, while the graph of

$$\frac{1}{x - y} = 0$$

is the null set.

The difficulty lies in the notion of factoring. When we speak of factoring a positive integer, we mean expressing it as the product of two smaller positive integers. When we speak of factoring a polynomial, we mean expressing it as the product of two polynomials each of lower degree than the given polynomial and having coefficients of some specified type (say rational numbers). There is no such agreement as to what it means to factor an arbitrary function. For our present purposes it is enough to say that we have a factorization of $f(x,y)$ if, for every (x,y) in the domain of f,

$$f(x,y) = g(x,y) \cdot h(x,y) .$$

Of course, this allows uninteresting factorizations like

$$x^2 + y^2 = 1 \cdot (x^2 + y^2)$$

but it excludes the sort of thing that got us into trouble above, since $x + y$ is defined for every x and y, while $\frac{1}{x - y}$ is not defined if $x = y$.

With this interpretation of "factor" we can state the principle referred to above.

THEOREM 6-1. If $f(x,y)$ has the factorization

$$f(x,y) = g(x,y) \cdot h(x,y) .$$

The graph of $f(x,y) = 0$ is the union of the graphs of $g(x,y) = 0$ and $h(x,y) = 0$.

Proof: The point (a,b) is on the graph of

$$f(x,y) = 0$$

if, and only if,

$$f(a,b) = 0 .$$

But

$$f(a,b) = g(a,b) \cdot h(a,b)$$

and hence

$$f(a,b) = 0$$

if, and only if

$$g(a,b) = 0$$

or

$$h(a,b) = 0$$

that is, if, and only if, (a,b) lies on the graph of

$$g(x,y) = 0$$

or the graph of

$$h(x,y) = 0 .$$

Example 10. The graph of

$$(y - x + 2)(x^2 + 4y^2 - 2x + 16y + 13) = 0$$

is made up of the graph of

$$y - x + 2 = 0$$

and the graph of

$$x^2 + 4y^2 - 2x + 16y + 13 = 0 .$$

The former is a straight line. If we rewrite the equation of the latter in the form

$$\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{1} = 1$$

we see that it is an ellipse, with center $(1,-2)$, symmetric about the lines

$$x = 1$$

and

$$y = -2$$

and with major and minor axes of lengths 4 and 2, respectively. Both graphs are sketched below.

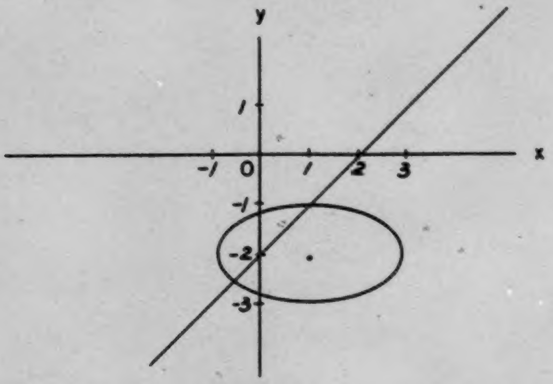


Figure 6-21

If we are given two parametric equations for a locus in a plane, there are two methods of sketching the locus (unless the equations are too complicated). We can eliminate the parameter between the two equations and graph the resulting equation in x and y, or we can choose some values of the parameter, compute the corresponding values of x and y, and draw a curve through the points thus determined. We illustrate both methods in the next example.

Example 11. Draw the graph of the parametric equations

$$x = 4t^2 - 2 \,, \quad y = 4t^4 \,. \tag{1}$$

Solution. First let's eliminate the parameter and graph the resulting equation. From the first equation we find that $2t^2 = \frac{x + 2}{2}$. Substituting this in the second equation gives

$$y = \frac{1}{4}(x + 2)^2 \,. \tag{2}$$

The graph of (2) is a parabola. It is sketched below.

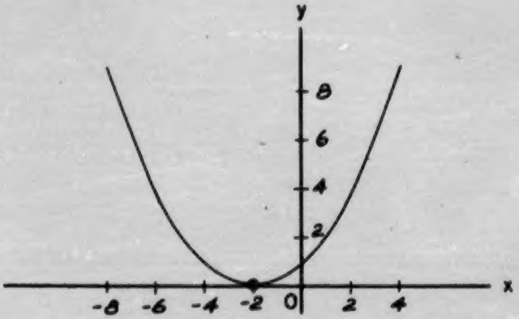


Figure 6-22

Now let's use the second method described above. The table below shows the results of our computations.

t	-2	-1	0	1	2
x	14	2	-2	2	14
y	64	4	0	4	64

We notice at once that we have found no values of x smaller than -2. It would be natural to jump to the conclusion that we had chosen the values of t foolishly, but that is not the explanation. Since $x = 4t^2 - 2$ and $4t^2 \geq 0$, it follows that $x \geq -2$ for every point on the graph. The trouble is that Equations (1) and (2) are simply not equivalent. The graph of (1) is half a parabola. It is the intersection of the graphs of (2) and the inequality $x \geq -2$. If you look back over our reasoning you will see it proves that the locus of (1) is contained in the locus of (2), but it does not prove they are identical.

Obviously the elimination of t was not as harmless an operation as it looked and we must study it more carefully. At a certain point we found from the first equation in (1) that $2t^2 = \frac{x+2}{2}$. Then we squared, getting the equation $4t^4 = \frac{(x+2)^2}{4}$. These two are not equivalent, since in the first, $x \geq -2$ while the second puts no restriction on x. This is no surprise since the same sort of thing comes up in the solution of equations involving radicals. In future we shall be careful not to square, or divide by zero, or do anything else of that sort when eliminating a parameter, and then perhaps we'll not get into trouble as we did above. Unfortunately it isn't that simple.

Example 12. What locus is represented by the parametric equations

(3) $$x = \sin t \qquad y = \sin t \text{ ?}$$

Solution. Eliminating t in the only sensible way gives the equation $y = x$. The graph of this is a line, while the locus of (3) is the segment determined by $(-1,-1)$ and $(1,1)$. Equations (3) are an analytic condition for a segment stated without inequalities.

There is no simple way out of this difficulty, and we end our discussion with the warning that when you eliminate the parameter from a pair of parametric equations for a curve, you must then check to see whether the locus of the resulting equation is the locus of the original pair of equations.

The nature of the parameter may impose certain natural restrictions or bounds on the values of the variables involved. In some problems we may wish to impose such restrictions, and in that case we have, not a difficulty, but a special tool. It is important that we learn the uses and limitations of our tools, so that we do not try to use a screwdriver to drive nails.

All the analytic conditions we have considered so far in this section have been equations. Our last two examples deal with inequalities.

Example 13. Disucss and sketch the locus of the inequality

$$2x - 3y + 4 < 0 .$$

Solution. We shall use simple arguments about inequalities. Suppose (x_0,y_0) is on the line $2x - 3y + 4 = 0$, so that $2x_0 - 3y_0 + 4 = 0$. Now consider a point (x_0,y_1), with $y_1 > y_0$. Then $3y_1 > 3y_0$ and $2x_0 - 3y_1 + 4 < 2x_0 - 3y_0 + 4 = 0$. Thus (x_0,y_1) is a point of the locus. Similarly, if $y_2 < y_0$, $2x_0 - 3y_2 + 4 > 0$ and (x_0,y_2) is not a point of the locus. Thus any point directly above a point of the line is in the locus, while any point directly below a point of the line is not. Therefore the locus is the half-plane indicated below.

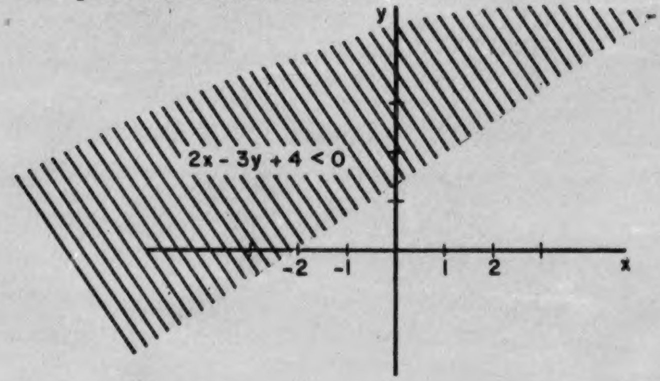


Figure 6-23

Example 14. Discuss and sketch the locus of the inequality

$$2x^2 - 8x - y + 7 \geq 0 . \tag{4}$$

Solution. By completing the square we can rewrite this inequality in the form

$$2(x - 2)^2 - y - 1 \geq 0 .$$

Now suppose $2(x_0 - 2)^2 - y_0 - 1 = 0$. If $y_1 \leq y_0$ then $2(x_0 - 2)^2 - y_1 - 1 \geq 0$. Thus if (x_0,y_0) is on the graph of the equation

$$2(x - 2)^2 - y + 1 = 0 \tag{5}$$

and $y_1 \leq y_0$, we see that (x_0,y_1) is a point of our locus. By a similar argument we can show that if $y_2 > y_0$, then (x_0,y_2) is not a point of our locus. Thus our locus is the set of points below or on the parabola represented by Equation (5). It, or rather some of it, is shaded in the sketch below.

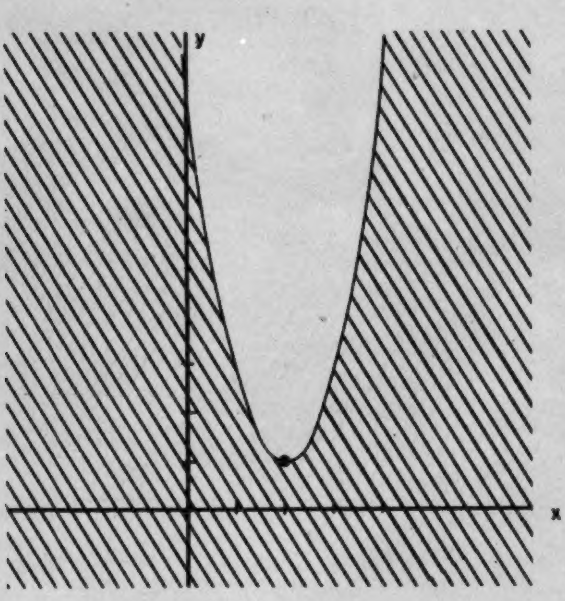


Figure 6-24

Exercises 6-3

In these exercises discuss and sketch the graphs of the conditions given. In your discussion you may find it useful to consider symmetry, extent, periodicity, intercepts, and asymptotes. When the condition is a pair of parametric equations, eliminate the parameter if you can, but be sure then to indicate any restrictions on the values of the variables.

1. $y = 2$
2. $y = -3$
3. $x = -1$
4. $x = 4$
5. $y = -x + 3$
6. $y = 2x - 1$
7. $x - 2y + 3 = 0$
8. $2x + 3y - 5 = 0$
9. $\frac{x}{2} - \frac{y}{3} = 1$

44

10. $\frac{x}{3} + \frac{y}{4} = 1$

11. $x = 1 - 2t$, $y = 2 + 3t$

12. $x = 2t$, $y = -2 - t$

13. $x^2 + y^2 - 4x + 2y + 4 = 0$

14. $x^2 + y^2 + 2x - 3 = 0$

15. $x^2 + y^2 + 2x - 2y + 2 = 0$

16. $y^2 = x(x - 2)(x - 3)$

17. $x^2 = (y + 1)(y - 1)(y - 4)$

18. $xy^2 - 2y - x = 0$

19. $y = \sin 2x$

20. $x = \sin y$

21. $y = 2 \sin x$

22. $x = \cos y$

23. $y = 1 + \cos x$

24. $y = \tan 2x$

25. $y = 2^x$

26. $y = 2^{-x}$

27. $y = 2^{x^2}$

28. $y = 3^{x^3}$

29. $y = \ln x$ (Note: This may also be written $y = \log_e x$.)

30. $y = \ln x^2$ (See above.)

31. $y = \log_2 x$

32. $x = t^2 + 1$, $y = 5t^2 + 4$

33. $x = \frac{1}{t}$, $y = 3t$

34. $x = 2 \cos \Phi$, $y = 2 \sin \Phi$

35. $x = 2 \cos \Phi$, $y = 4 \sin \Phi$

36. $x = 3 \cos^3 \Phi$, $y = 3 \sin^3 \Phi$

37. $x = \sin^2 \Phi$, $y = \cos^2 \Phi$

38. $x = \sec^2 \theta$, $y = \tan^2 \theta$

39. $y > x^2$

40. $\frac{x^2}{9} + \frac{y^2}{4} < 1$

41. $y^2 - 2x - 4y + 2 < 0$

42. $x^2 + y^2 + 4x + 6y + 9 \geq 0$

43. $y^2 = x^3$

44. $x^3 + xy^2 - 4y^2 = 0$

45. $x^3 + xy^2 - 3x^2 + y^2 = 0$

46. $x^2y + 4y - x = 0$

47. $x^4 + y^4 = a^4$

6-4. Graphs and Conditions (Polar Coordinates)

In this section we discuss the problem of sketching the graphs of analytic conditions in polar coordinates. The most important such conditions are equations, and we shall confine our attention to this case except for a few exercises.

The most straightforward way to draw the graph of an equation in polar coordinates is to plot a number of points of the locus and draw a curve through them. If the equation has the form $r = f(\theta)$, we can construct a table giving the values of r corresponding to a number of values of θ. No matter how many points we plot, there always remains the question of how the curve behaves elsewhere, that is, between the points we have plotted. If the equation is not too complicated, we can get a good deal of information by studying the functions involved.

As was the case for equations in rectangular coordinates, we can often get useful information about the curve by considering symmetry and extent. Asymptotes of curves given by equations in polar coordinates are not easy to find from the equations, and we shall not discuss the problem. However, if the curve has a fairly simple equation in rectangular coordinates, we may be able to find its asymptotes by studying that.

As you know, given a polar coordinate system in a plane, each point has infinitely many pairs of coordinates. This fact gives rise to certain difficulties that we have already met in Chapter 5 but we now consider them in greater detail. As in the previous section we shall develop additional theory and useful methods of approach in our discussion of a number of examples.

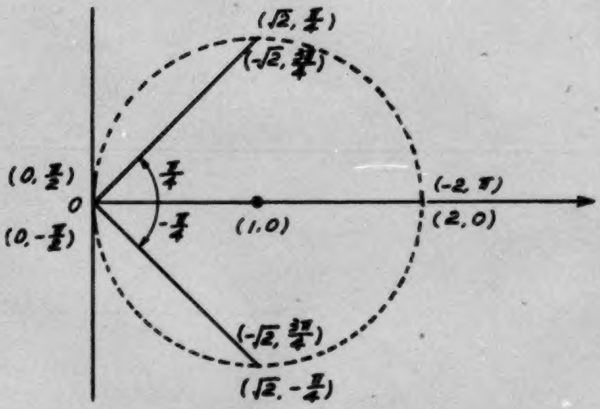
Example 1. Sketch and discuss the graph of the equation $r = 2\cos\theta$.

Solution. Strictly speaking, we should state explicitly that r and θ are to be interpreted as polar coordinates. We shall not do so in the rest of this section, since there is no danger of ambiguity.

Since $|\cos\theta| \leq 1$ for all θ , the graph is bounded. Since $\cos(-\theta) = \cos\theta$ for all θ , if the point (r_0, θ_0) is on the graph, so is the point $(r_0, -\theta_0)$. Thus the graph is symmetric with respect to the line containing the polar axis. It is also symmetric with respect to the point $(1,0)$, but it is much easier to show this by using an equation in rectangular coordinates for the locus. The table below shows the values of r corresponding to several values of θ . The cosine function has period 2π , so any θ-interval of length 2π will do.

θ	0	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{2}$	$\pm\frac{3\pi}{4}$	π
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2

The graph is sketched below.



$(\sqrt{2}, \frac{\pi}{4})$
$(-\sqrt{2}, \frac{3\pi}{4})$
$(0, \frac{\pi}{2})$
$\frac{\pi}{4}$
0
$-\frac{\pi}{4}$
$(0, -\frac{\pi}{2})$
$(1,0)$
$(-2, \pi)$
$(2,0)$
$(-\sqrt{2}, \frac{3\pi}{4})$
$(\sqrt{2}, -\frac{\pi}{4})$

Figure 6-25

It looks like a circle (probably because it was drawn with a compass), but all we know so far, even if we make use of our knowledge of the cosine function, is that it is roughly circular.

That the graph really is a circle can be proved as follows. The graph of $r^2 = 2r \cos \theta$ is the same as the graph of $r = 2 \cos \theta$. For the only points that might be on the former but not on the latter are points with $r = 0$, and the origin, which is on the latter, is the only such point. If we take a rectangular coordinate system with its axes in the usual positions with respect to the polar axis, we find that the graph has the equation

$$x^2 + y^2 = 2x .$$

Example 2. Sketch and discuss the graph of the equation $r = \sin 3\theta$.

Solution. This graph, too, is bounded, since $|\sin 3\theta| \leq 1$ for all θ. Whether there is a point or line about which the graph is symmetric is not obvious from the equation, so we postpone the discussion of symmetry till we have sketched the graph. It will prove nothing but it will suggest what is probably true. The table below shows the values of r corresponding to a number of values of θ. If we needed a fairly accurate graph of the equation we would have to consider more values of θ, but since we know how $\sin 3\theta$ varies with θ, this table will do.

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
r	1	0	-1	0	1	0	-1	0	1	0	-1

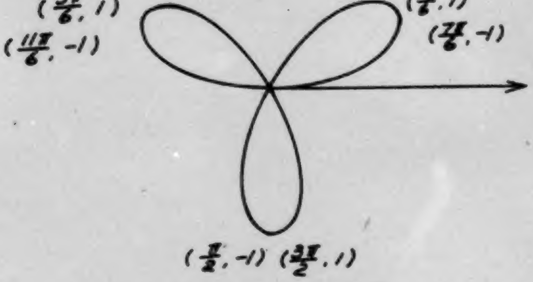


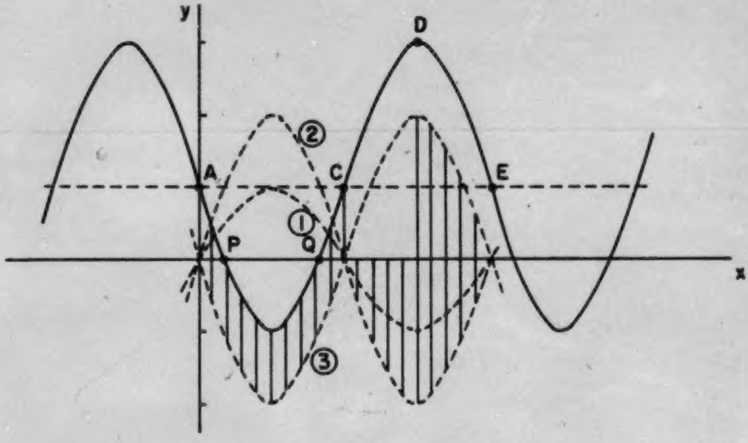
Figure 6-26

The sketch suggests there is symmetry about each of the lines $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$, and $\theta = \frac{3\pi}{2}$. Let us check the first of these conjectures. If we wish to compare $f(\frac{\pi}{6} + \alpha)$ and $f(\frac{\pi}{6} - \alpha)$ we obtain in the first case $r = \sin 3(\frac{\pi}{6} + \alpha)$ and in the second case $r = \sin 3(\frac{\pi}{6} - \alpha)$. These become $r = \sin(\frac{\pi}{2} + 3\alpha)$, and $r = \sin(\frac{\pi}{2} - 3\alpha)$; which in turn become $r = \cos 3\alpha$ and $r = \cos 3\alpha$. The identity of these equations establishes the symmetry we were checking. The same method can be used to deal with the other lines. The graph is not symmetric about any point, but we shall not prove this.

Example 3. Sketch and discuss the graph of the equation $r = 1 - 2 \sin \theta$.

Solution. Once more the graph is bounded, and we postpone the discussion of symmetry until below.

This time we shall sketch the graph without making a table, introducing first an auxiliary graph of a kind that is often useful in graphing polar equations. This auxiliary graph is the graph of the equation $y = 1 - 2 \sin x$, drawn on a plane with a rectangular coordinate system. We have learned to do this readily by the addition and multiplication of ordinates, as shown in Section 6-3, and illustrated below for the values $0 \leq x \leq 2\pi$. For the purpose of illustrating certain details of the discussion we will sometimes use different scales on the axes in the graphs in this section.



We suggest the following sequence:

(1) Sketch the familiar curve ① : $y = \sin x$.

(2) Expand ① away from the x-axis to get ② : $y = 2 \sin x$.

(3) Reflect ② in the x-axis to get ③ : $y = -2 \sin x$.

(4) Raise ③ 1 unit to get our graph: $y = 1 - 2 \sin x$.

We now use this graph of the equation $y = 1 - 2 \sin x$ to give us coordinates of points of the polar graph of $r = 1 - 2 \sin \theta$, and obtain the polar graph given in Figure 6-28.

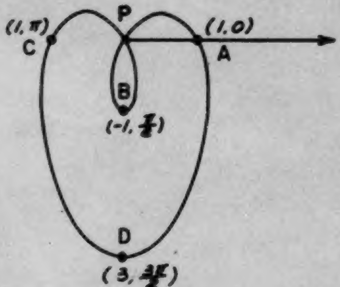


Figure 6-28

This curve is called a limaçon. We have indicated with the same letters corresponding points on the two graphs. Note that the lack of a unique polar representation of a point is shown in the fact that points P and Q of Figure 6-27 (and infinitely many more not shown) all correspond to point P of Figure 6-28. Also, points A and E of Figure 6-27 (and infinitely many more not shown) all correspond to point A of Figure 6-28. The inverted arch below the x-axis of Figure 6-27 corresponds to the small-inside loop of Figure 6-28.

Figure 6-28 suggests that the graph is symmetric about the line through the pole perpendicular to the polar axis, that is, the line for which one equation is $\theta = \frac{\pi}{2}$. We check this by comparing $f(\frac{\pi}{2} - \alpha)$ and $f(\frac{\pi}{2} + \alpha)$. In the first case $r = 1 - 2 \sin(\frac{\pi}{2} - \alpha)$ and in the second case $r = 1 - 2 \sin(\frac{\pi}{2} + \alpha)$. In both cases we obtain from familiar trigonometric relationships $r = 1 - 2 \cos \alpha$. which means that the two cases give equivalent equations, and the symmetry is proved.

Finally, the related polar equation is $r = -(1 - 2\sin(\theta + \pi)) = -(1 + 2\sin\theta)$. To show that the polar graph of this equation is the same limaçon as the one we obtained in Figure 6-28, we use a method similar to the method of addition of ordinates for graphs in rectangular coordinates. The method, called addition of radii, which may be new to you, is useful in sketching certain new graphs related to familiar ones.

We have seen earlier that the polar graph of $r = 2\sin\theta$ is a circle of radius 1 , with its center at $(1, \frac{\pi}{2})$ indicated as ① in Figure 6-29(a). Consider a number of rays drawn from 0 to points of this circle, $\overrightarrow{OP}_1$, $\overrightarrow{OP}_2$, $\overrightarrow{OP}_3$, ... Find points Q_1 , Q_2 , Q_3 , ... on these respective rays so that $d(P_1,Q_1) = d(P_2,Q_2) = d(P_3,Q_3)$, ... , $= 1$, as shown in Figure 6-29(a) , which shows the graph of $r = 1 + 2\sin\theta$.

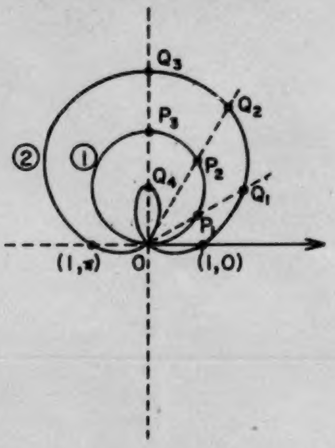


Figure 6-29(a)

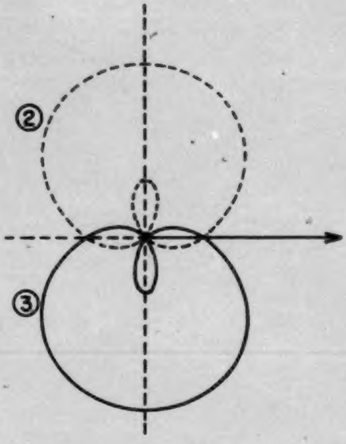


Figure 6-29(b)

Note that when $\pi < \theta < \frac{3\pi}{2}$ we have $0 > 2\sin\theta > -2$, therefore $1 > (1 + 2\sin\theta) > -1$, and the Q points of Figure 6-29(a) are on the right half of the inside loop of the graph. In the same way when $\frac{3\pi}{2} < \theta < 2\pi$ we get the rest of the inside loop.

Thus the locus of all the Q points is the graph marked ② which is a limaçon whose polar representation is $r = 1 + 2\sin\theta$. This process of using the P points to find the Q points and the graph ② is called the addition of radii.

Since we want the graph of $r = -(1 + \sin\theta)$ we now find the symmetric image of ② with respect to the pole. It is graph ③ which we recognize as the same limaçon as in Figure 6-28.

Example 4. Discuss and sketch the graph of the equation $r = \dfrac{1}{1 + \sin\theta}$.

Solution. This graph is not bounded, since r can be made arbitrarily large by picking θ so that $\sin\theta$ is sufficiently close to -1. By the method used in Examples 2 and 3, we find the graph is symmetric about the line $\theta = \frac{\pi}{2}$. It can be sketched from the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	2π
r	1	$\frac{1}{2}$	1	3.4	7.5	Undefined	7.5	3.4	1

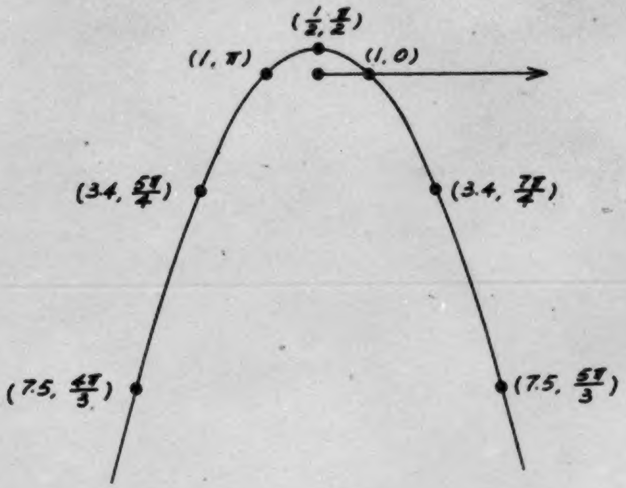


Figure 6-30

The sketch suggests the graph may be a parabola. That it is may be shown as follows. The equation

$$r = \frac{1}{1 + \sin\theta}$$

is equivalent to the equation

$$r + r \sin \theta = 1 .$$

If we introduce a rectangular coordinate system with its axes located as usual, the graph has the equation

$$\sqrt{x^2 + y^2} = 1 - y .$$

This is an equation of the parabola consisting of all points as far from the origin as they are from the line $y = 1$.

Exercises 6-4

In each of the exercises below, discuss and sketch the graph of the condition given. In your discussion, consider whatever geometric properties you can infer from the equations. Write the related polar equation for each. If you can, find a condition in rectangular coordinates for the same locus and identify the locus.

1. $r = 3$
2. $r = -2$
3. $\theta = \frac{\pi}{6}$
4. $\theta = -\frac{3\pi}{2}$
5. $r = 3 \sin \theta$
6. $r = \sin 2\theta$
7. $r = \cos 2\theta$
8. $r = \sin 5\theta$
9. $r \cos \theta = -3$
10. $r \cos (\theta - \frac{5\pi}{6}) = 3$
11. $r = \frac{3}{1 - \cos \theta}$
12. $r = \frac{9}{4 - 5 \cos \theta}$
13. $r = 2(1 + \sin \theta)$
14. $r = 2 \tan \theta$. (There are vertical asymptotes; try to find them.)
15. $r = \frac{4}{\theta}$
16. $r = 2 \cos \theta - 1$
17. $r = 2 - 3 \cos \theta$
18. $r = 2 + \sin \theta$

19. $r^2 = \cos 2\theta$

20. $r^2 = 4 \sin 2\theta$

21. $r = 4 \tan\theta \sec\theta$

22. $r = 2(1 + \sin^2\theta)$

23. $r = \dfrac{5}{1 + \cos\theta}$

24. $r \leq 2$

25. $|r| \leq 2$

26. $2 < r < 3$

27. $0 \leq \theta \leq \frac{\pi}{4}$

28. $0 \leq \theta \leq \frac{\pi}{4}$, $r \geq 0$

6-5. Intersections of Graphs (Rectangular Coordinates)

The intersection of two sets is the collection of objects that belong to both the sets. Now the graph of the equation $f(x,y) = 0$ is the set of points whose coordinates satisfy the equation, i.e. $\{(x,y) : f(x,y) = 0\}$. Hence the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is the set of points whose coordinates satisfy both equations, i.e. $\{(x,y) : f(x,y) = 0$ and $g(x,y) = 0\}$. If f and g are linear functions, the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is the set of points which lie on two lines, in other words the intersection of the two lines. In general, the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is found by solving the two equations simultaneously.

Example 1. The intersection of the lines with equations $x - 2y - 1 = 0$ and $x + y = 2$ is the point $(\frac{5}{3}, \frac{1}{3})$.

Example 2. The intersection of the lines with equations $x - 2y - 1 = 0$ and $2x - 4y - 3 = 0$ is the null set. In other words, the lines are parallel.

Example 3. The intersection of the graphs of $y = \sin x$ and $y = \cos x$ is a bit harder to find. At each point (x,y) where the curves intersect we have $\sin x = \cos x$. Thus $x = \frac{\pi}{4} + k\pi$, where k is an integer. Then

$y = \frac{\sqrt{2}}{2}$ when k is even, $y = -\frac{\sqrt{2}}{2}$ when k is odd. This last statement can be written more compactly in a form frequently used by mathematicians:

$y = (-1)^k \frac{\sqrt{2}}{2}$, where k is an integer.

Example 4. The intersection of the graphs of $x - y + 3 \leq 0$ and $2x - xy + 4 \geq 0$ is the set of points on or above the line $x - y + 3 = 0$ and on or below the line $2x - y + 4 = 0$. It is the doubly shaded area in the figure below, and its boundary along parts of the lines.

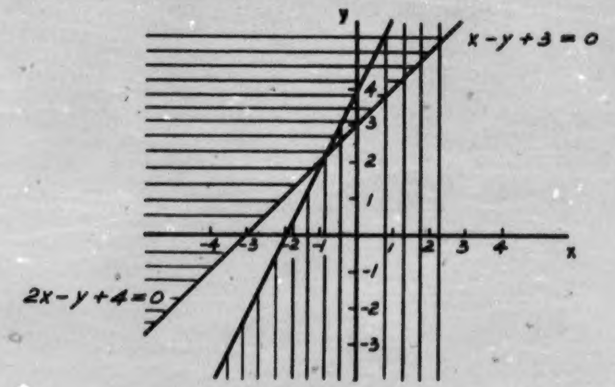


Figure 6-31

The problem of finding the intersection of two graphs can be very complicated, and we shall not spend much more time on it here. However, there is another example which is of interest.

Example 5. Find the intersection of $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 2x + 2y - 2 = 0$. We could consider the first equation as a quadratic equation in y and use the quadratic formula to express y in terms of x. We could get $y = 2 \pm \sqrt{8 + 2x - x^2}$. We could then substitute this in the second equation and solve for x. (Carry the work a bit further so you will appreciate the difficulties.)

This problem can be solved much more easily by using the principle of linear combination, which you studied in algebra. The system

$$(1) \quad \begin{aligned} x^2 + y^2 - 2x - 4y - 4 &= 0 \\ x^2 + y^2 + 2x + 2y - 2 &= 0 \end{aligned}$$

is equivalent to the system

$$\begin{aligned} a(x^2 + y^2 - 2x - 4y - 4) + b(x^2 + y^2 + 2x + 2y - 2) &= 0 \\ x^2 + y^2 + 2x + 2y - 2 &= 0 \end{aligned} \tag{2}$$

as long as $a \neq 0$. If $a = -1$ and $b = 1$, the second system becomes

$$\begin{aligned} 4x + 6y + 2 &= 0 \\ x^2 + y^2 + 2x + 2y - 2 &= 0 \, . \end{aligned} \tag{3}$$

Now the first equation in (3) is linear. Using it, we can express y in terms of x , substitute the result in the second equation, and have left nothing worse than a quadratic equation in x . The points of intersection are $(1,-1)$ and $(-\frac{23}{13},\frac{11}{13})$.

This solution has a geometric interpretation which is worth investigating.

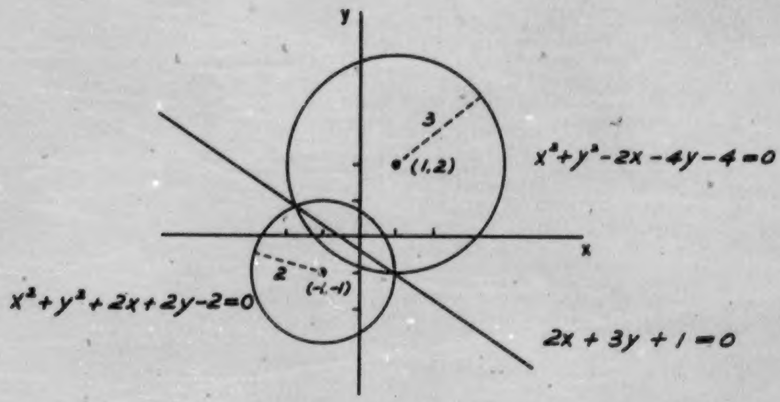


Figure 6-32

The graphs of the equations in (1) are circles. (How can you check this?) They are shown above. Now the graph of the first equation in (3) is a line and that equation is a special case of the first equation in (2). But if the coordinates of a point satisfy the two equations in (1) , they clearly satisfy the first equation in (2) , no matter what a and b are. Thus the graph of the first equation in (3) passes through all points of intersection of the two circles and must be the line containing the common chord, which is shown in the sketch above. If $a \neq -b$ which implies that a and b are not both zero, the first equation in (2) is that of a circle passing through the points of intersection of the two original circles. (As a matter of fact, each such circle may be obtained by some choice of a and b . Can you prove this?)

This result can be generalized. If $f(x,y) = 0$ and $g(x,y) = 0$ are equations of two loci, then the locus of $af(x,y) + bg(x,y) = 0$ contains the intersection of the two original loci. For suppose (x_0,y_0) lies on the original loci. Then $f(x_0,y_0) = 0$, $g(x_0,y_0) = 0$, and hence $af(x_0,y_0) + bg(x_0,y_0) = 0$. (This is true, though not very interesting, even when $a = b = 0$.)

Exercises 6-5

In each of the exercises below, find the intersection of the loci determined by the conditions given. Use both algebraic and geometric methods.

1. $x = 2$, $x - 2y = 2$
2. $x - y + 1 = 0$, $2x + y - 7 = 0$
3. $x + y - 1 = 0$, $2x + y = 0$
4. $x - 2y + 3 = 0$, $2x + y - 2 = 0$
5. $x - 2y + 3 = 0$, $2x - 4y + 5 = 0$
6. $x^2 + y^2 = 4$, $y = 2x$
7. $x^2 + y^2 = 2$, $x + y = 0$
8. $x^2 + y^2 - 2x + 4y + 5 = 0$, $3x + y - 1 = 0$
9. $x^2 + y^2 + 2x + 2y - 2 = 0$, $\frac{x}{4} + \frac{y}{5} = 1$
10. $y^2 = 4x$, $x - 2y + 3 = 0$
11. $4x^2 - 3y^2 = 1$, $x - y = 0$
12. $x^2 + 2y^2 = 4$, $x - y - 1 = 0$
13. $x^2 + y^2 = 11$, $x^2 + y^2 - 2x - 8 = 0$
14. $x^2 + y^2 = 15$, $2x^2 + y^2 = 24$
15. $xy^2 - xy - 4y + 4 = 0$, $y = x$
16. $x^2 + y^2 = 2$, $y = x^2$
17. $y - x^2 > 0$, $y - x - 1 < 0$

18. $x^2 + y^2 \leq 4$, $x - y^2 > 0$

19. $x + 2y + 3 < 0$, $3x - y + 5 > 0$, $2x - 3y + 1 < 0$

6-6. Intersection of Loci (Polar Coordinates)

In the previous section we discussed the intersection of loci given by equations in rectangular coordinates. The method we used works for loci determined by equations in polar coordinates, but, as we shall see, there are added complications. Let us take up first a simple case.

Example 1. Consider the graphs of $r = 1$ and $r = 2\cos\theta$. They are the circles shown below.

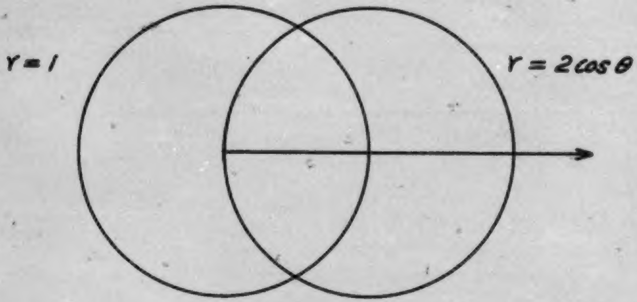


Figure 6-33

Solving the equations simultaneously we get $2\cos\theta = 1$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. (There are infinitely many other solutions of the equations, but since the sine and cosine functions have period 2π , we need consider only solutions with $0 \leq \theta < 2\pi$.) Of course, $r = 1$. This is consistent with our sketch.

__Example 2__. Now consider the equations $r = 2\cos\theta$ and $r = 2\sin\theta$. Once more their graphs are circles, which are shown in the figure below.

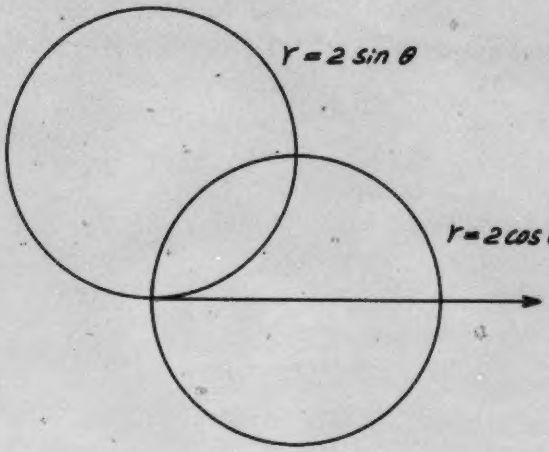


Figure 6-34

There appear to be two points of intersection. Let us solve the two equations simultaneously and compare our answer with the figure. Setting $2\cos\theta = 2\sin\theta$ we find $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. (As before, we need consider only solutions with $0 \leq \theta \leq 2\pi$.) The first gives $r = \sqrt{2}$, the second $r = -\sqrt{2}$. We have not, however, found the two points of intersection shown in the figure. We have found two sets of polar coordinates for the same point. This reminds us once more that while a rectangular coordinate system in a plane is a one-to-one correspondence between the points in the plane and the ordered pairs of real numbers, every point in the plane has infinitely many different pairs of polar coordinates.

This is also the source of our other difficulty. Clearly the pole lies on both curves, but our algebraic method did not find this intersection. The trouble is that the coordinates $r = 0$, $\theta = \frac{\pi}{2}$ satisfy the first equation but not the second, while the coordinates $r = 0$, $\theta = 0$ satisfy the second but not the first. Both pairs, of course, represent the pole, whose coordinates require special comment. If P is any point other than the pole, its coordinates, $(r, \theta + 2n\pi)$, allow infinitely many, but not all numbers as second coordinate. For the pole, however, the coordinates $(0,\theta)$ allow __any__ number as a possible replacement for θ. Geometrically this means that, if there is any θ for which $r = f(\theta)$ becomes zero, the graph must contain the pole. We have already found in this example that $(0,\frac{\pi}{2})$ satisfies the first equation, and $(0,0)$ the second, which means that the pole lies on both graphs and is therefore a point of intersection.

This leads to a small but important caution when finding intersections of polar graphs of $r = f(\theta)$, and $r = g(\theta)$. Check first to see if each graph contains the pole by seeing if there is any θ for which $r = f(\theta)$ equals zero, or any ϕ for which $r = g(\phi)$ equals zero. If both conditions can be satisfied, then, whether or not $\theta = \phi$, both graphs contain the pole, which is therefore an intersection point. Then you can proceed with the usual simultaneous solution of the two equations.

Example 3. Find the points of intersection of the graphs of

$$r = \frac{1}{2 + 2\cos\theta} \quad \text{and} \quad r = 2\cos\theta + 1 .$$

Solution. These graphs, which are related to some we have discussed earlier, are shown below. The pole is on the second graph but not the first, hence is not a point of intersection.

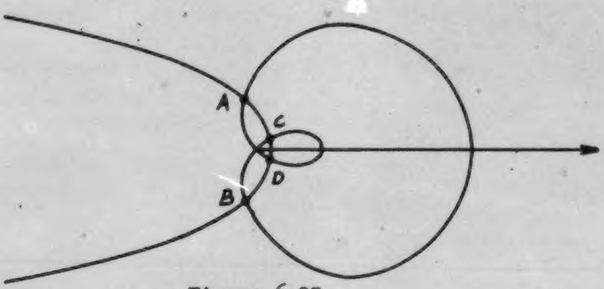


Figure 6-35

There appear to be four points of intersection.

Now let us solve the two equations simultaneously. Setting the expressions for r in the two equations equal to each other, we get

$$\frac{1}{2 + 2\cos\theta} = 2\cos\theta + 1 .$$

Simplifying, we get

$$4\cos^2\theta + 6\cos\theta + 1 = 0$$

from which we find that

$$\cos\theta = \frac{1}{4}(-3 \pm \sqrt{5})$$

or

$$\cos\theta \approx -1.31 \quad \text{or} \quad -.19 \;.$$

The first is a perfectly good root of the quadratic equation for $\cos\theta$, but it is not a possible value for $\cos\theta$. (Why not?) From a table of values of the trigonometric functions we find that if $\cos\theta \approx -.19$, then

$$\theta \approx 101^\circ \quad \text{or} \quad \theta \approx 259^\circ \;.$$

Then

$$r \approx .62 \;.$$

It is clear that we have found the points A and B of the figure, but what about C and D ? It is not too hard to guess the answer if we remember that a polar graph may have other analytic representations. In our algebraic solution we merely equated two of the infinitely many equivalent polar equations available for each curve. Fortunately we need not try them all; for the purposes of the course we can always find all the intersections of two polar graphs from the simultaneous solution of an equation of one of them with both of the related polar equations of the other. The limaçon $r = 2\cos\theta + 1$ has the related polar equation $r = -(2\cos(\theta + \pi) + 1)$ or $r = 2\cos\theta - 1$. If we now solve simultaneously the equations

$$r = \frac{1}{2 + 2\cos\theta} \quad \text{and} \quad r = 2\cos\theta - 1$$

we get the coordinates of points C and D in our figure. They turn out to be approximately, $(.30, 49^\circ)$ and $(.30, 311^\circ)$.

The difficulty is not a simple one, so we shall take another look at it. Consider:

$$\begin{cases} (.62, 101^\circ) \\ (-.62, 281^\circ) \end{cases} \qquad\qquad \begin{cases} r = 2\cos\theta + 1 \\ r = 2\cos\theta - 1 \end{cases}$$

We have two pairs of coordinates for the same point, and two equations for the same curve. The first pair of coordinates satisfies the first equation but not the second and the second pair of coordinates satisfies the second but not the first. This situation should occasion not anxiety but care, and is entirely consistent with our definition of the polar graph of an equation as the set of points each of which has some pair of coordinates that satisfy it.

Exercises 6-6

In each of the exercises below, find the intersection of the loci determined by the conditions given. Write the related polar equation for each, to make sure you find all points of intersection. Sketch both loci, as a check on your algebra.

1. $r = \frac{2}{1 + \cos\theta}$, $\theta = 30^\circ$
2. $r = \frac{4}{1 + \sin\theta}$, $\theta = 135^\circ$
3. $r = 2\cos\theta$, $r = 2\sin\theta$
4. $r = \cos\theta$, $r = 1 - \cos\theta$
5. $r = \cos\theta$, $r = \sin 2\theta$
6. $r = 1 - \sin\theta$, $4r\sin\theta = 1$
7. $r = 1 + \cos\theta$, $r = \frac{1}{1 - \cos\theta}$

6-7. Families of Curves.

In Section 6-5 we mentioned the collection of lines through the intersection of two lines and the collection of circles (and the line) through the intersections of two circles. These are examples of what are called families of curves. The collection of all circles in a plane and the collection of all tangents to a parabola are other examples. In this section we shall proceed a bit further with this topic.

If a and b are not both zero, then

$$a(x - y + 3) + b(3x - y + 7) = 0 \tag{1}$$

is an equation of a line through the intersection, P , of

$$x - y + 3 = 0 \text{ and } 3x - y + 7 = 0 .$$

Can we choose a and b so that the line is vertical? Yes. For if we let $a = 1$ and $b = -1$, the equation becomes

$$-2x - 4 = 0$$

or

$$x = -2 .$$

This is one method you learned in algebra for solving pairs of linear equations in two unknowns. In a similar way we could find the horizontal line through the intersection, which is equivalent to finding the y-coordinate of P. It turns out that $P = (-2,1)$.

Every line through $(-2,1)$ may be obtained by picking a and b suitably. For the slope of (1), if it has one, is $\frac{a + 3b}{a + b}$. If $a = -b$, then (1) has no slope, a fact we noted above in case $a = 1$, $b = -1$. And for any real number m, a and b may be chosen so that

$$\frac{a + 3b}{a + b} = m .$$

(This is not obvious. Can you prove it?)

Let us look at this family of lines from another point of view. The line through $(-2,1)$ with slope m has an equation

$$y - 1 = m(x + 2) . \tag{2}$$

For each real value of m we get a line, and different values of m give different lines. Thus, (2) is almost the same family as (1), the only difference being that the line $x = -2$, since it has no slope, is not a member of (2).

Among the members of the family (2) there should be two which are tangent to the circle $x^2 + y^2 = 1$. (One of them is obvious, but let's solve the problem as though we did not know one answer.) Intuitively, it is clear that a tangent to a circle is a line which intersects the circle in only one point. Let us solve (2) simultaneously with the equation of the circle, and then try to pick m so that there is only one solution. From (2),

$$y = mx + 2m + 1 .$$

Substituting this in $x^2 + y^2 = 1$ we get

$$x^2 + (mx + 2m + 1)^2 = 1$$

or

$$x^2 + m^2x^2 + 4m^2 + 1 + 4m^2x + 2mx + 4m = 1$$

or

$$(1 + m^2)x^2 + (4m^2 + 2m)x + 4m^2 + 4m = 0$$

This quadratic will have only one root (that is, a double root) if, and only if, its discriminant is zero. The discriminant turns out to be $-4m(3m + 4)$, which is zero if, and only if, $m = 0$ or $m = -\frac{4}{3}$.

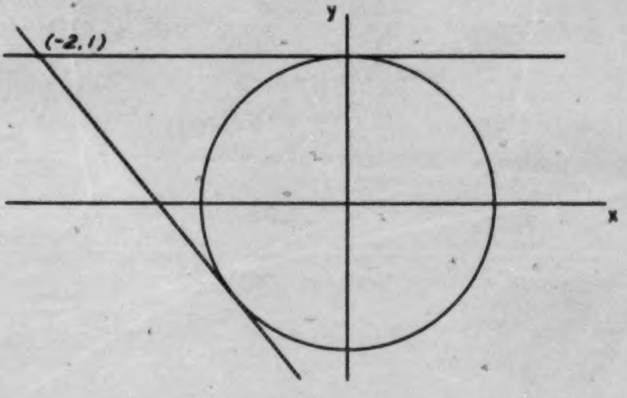


Figure 6-36

The figure shows the tangent lines for each case. Their equations are $y - 1 = 0$, and $4x + 3y + 5 = 0$.

Let us use the same method to find the family of tangents to the parabola $y = x^2$. Let (a,a^2) be any point on the parabola. The family of all but one of the lines through this point can be represented by the equation

$$y - a^2 = m(x - a).$$

(Which one is missing?) Expressing y in terms of a, m, and x, and substituting the result in the equation $y = x^2$, we get

$$x^2 - mx + ma - a^2 = 0.$$

This equation has a double root if, and only if, $m^2 - 4(ma - a^2) = 0$, i.e. if, and only if, $m = 2a$.. Thus the slope of the tangent to $y = x^2$ at (a,a^2) is $2a$, and the family of lines tangent to the parabola can be represented by the equation

$$y - a^2 = 2a(x - a)$$

or, in somewhat simpler form

$$y = 2ax - a^2 . \tag{3}$$

The "a" in (3) above is called a parameter. (The word was used earlier in the text in a different sense. That is, in a way, unfortunate, but both uses are very common.) It is difficult to define that word, but you must understand how "a" is used here. We might say "Let $\underline{a}$ be any real number. Then (3) is an equation of the tangent to $y = x^2$ at (a,a^2)." Here we are thinking of $\underline{a}$ as a fixed, but undetermined, real number. On the other hand, when we say that (3) represents the family of all tangents to the parabola $y = x^2$, we mean that each tangent to the parabola has an equation obtained by assigning a suitable real value to $\underline{a}$, and each equation so obtainable is an equation of a tangent to the parabola. In other words, (3) is an ingenious way of writing infinitely many equations in a small space.

You have considered many other families of curves in earlier courses, whether you used this phrase or not. The equation $Ax + By + C = 0$ represents the family of all lines in a plane. The equation $y = mx + b$ represents the family of all lines which have slopes, that is, all lines which are not perpendicular to the x-axis. The equation $xy = k$ represents the family of all rectangular hyperbolas with the coordinate axes as their asymptotes (and the two axes themselves, obtained by setting $k = 0$ and sometimes called a degenerate hyperbola). The equation $(x - h)^2 + (y - k)^2 = r^2$ represents the family of all circles in a plane (and the point (h,k), obtained by setting $r = 0$ and sometimes called a point circle).

Sometimes it is useful to consider a family of curves and select from it those which have some additional property. For example, at one point in the discussion above we considered the family of lines which pass through a point of $y = x^2$, and then selected from this family the member having the additional property of being tangent to the parabola. Let's consider an analogous problem.

The family of all the circles in the plane can be represented by the equation

$$(x - h)^2 + (y - k)^2 = r^2 .$$

The center of each such circle is at (h,k). Which members of the family are tangent to both axes? If a circle is tangent to both axes its center is on the line $y = x$ or on the line $y = -x$. The family of circles with centers on the line $y = x$ can be represented by the equation

$$(x - h)^2 + (y - h)^2 = r^2 .$$

Such a circle will be tangent to both axes if, and only if, $r = |h|$ or $r^2 = h^2$. Thus the family of circles lying in the first or third quadrant and tangent to both axes can be represented by the equation

$$(x - h)^2 + (y - h)^2 = h^2 .$$

An equation representing those in the second or fourth quadrant can be found in a similar way.

Exercises 6-7

In each of the first 13 exercises, find an equation representing the family of curves described.

1. All vertical lines.
2. All horizontal lines.
3. All nonvertical lines through $(2,-1)$.
4. All nonvertical lines.
5. All circles with center $(-1,2)$.
6. All circles with radius 4.
7. All parabolas with vertices at the origin and axes horizontal.
8. All lines parallel to $3x - 4y + 5 = 0$.
9. All lines perpendicular to $2x + y - 3 = 0$.
10. All lines tangent to the circle $x^2 + y^2 = 25$.
11. All lines that do not meet the circle $x^2 + y^2 = 25$.
12. All circles of radius 6 which go through the origin.
13. All circles of radius 1 such that the origin is not a point of the circle or its interior.
14. Find an equation of the line through the intersections of the lines $x - y + 6 = 0$ and $2x - y = 0$ and having x-intercept equal to 3.
15. Find an equation of the line through the intersection of $x + y - 4 = 0$ and $2x - y + 8 = 0$ and having slope 1.

16. Find an equation of the line passing through the intersection of the lines $x + y + 1 = 0$ and $x - 3y + 2 = 0$, and having no slope.

17. Find an equation of the line through the intersection of the lines $x - 2y + 3 = 0$ and $x + 3y - 2 = 0$ and the point $(1,1)$, without finding the intersection of the two lines.

18. Find an equation of the family of circles through the intersections of the circles $x^2 + y^2 - 2x - 35 = 0$ and $x^2 + y^2 + 2x + 4y - 44 = 0$, without finding the intersections of the two circles.

19. Find an equation of the line through the intersection of the lines $2x + 5y - 10 = 0$ and $3x - y + 19 = 0$ and perpendicular to the second of these lines.

20. Find an equation of the line through the intersection of $x + y - 4 = 0$ and $x - y + 2 = 0$ and parallel to $3x + 4y + 7 = 0$.

21. Find equations of all lines passing through the intersection of $5x - 2y = 0$ and $x - 2y + 8 = 0$ and cutting from the first quadrant triangles whose areas are 36.

22. Find equations of all lines through the intersection of $y - 10 = 0$ and $2x - y = 0$ which are 5 units from the origin.

6-8. Summary.

We have explored in some detail in this chapter the relations between the geometric properties of a set of points and the algebraic properties of its analytic representation. It was convenient to discuss the geometric properties under the headings of symmetry, extent, periodicity, intercepts, and asymptotes. We paid particular attention to the special situations that arise in polar coordinates from the lack of uniqueness in the correspondence between points and their polar coordinates, and the consequent lack of uniqueness in the correspondence between curves and their analytic representations.

Our discussion considered relationships between graphs and their conditions, first in rectangular and then in polar coordinates. We developed several useful techniques, notably the method of sketching a graph by addition and multiplication of ordinates in rectangular graphs, and by addition of radii in polar graphs.

These techniques were then applied to pairs of graphs and their intersections, and the corresponding pairs of analytic representations and their simultaneous solutions. We investigated in some detail the difficulties that arise here with polar coordinates and found the concept of related polar equations particularly useful in these cases.

Our consideration of more than two graphs at a time was confined to collections of graphs related by some common feature. These are called families of graphs, and we developed some useful concepts in defining such a family, and then selecting a particular member of it to fit some special requirement.

In our next chapter we sharpen our focus and discuss particularly a certain classification of graphs and their equations. These, the conic sections, have a valid claim to our special attention, both because they have been extensively studied for over 2000 years and because they have important and interesting application in many aspects of our lives today.

Chapter 6 - Review Exercises

1. Find the locus of the midpoint of all segments parallel to the x-axis, and terminated by the lines $x + y - 8 = 0$, $2x - y - 1 = 0$.
2. Find the locus of the midpoint of all segments parallel to the y-axis and terminated by the lines $x + y - 8 = 0$, $2x - y - 1 = 0$.
3. If $A = (-4,0)$ and $B = (4,0)$ find an equation for the locus of $P = (x,y)$ if:
   - (a) $d(P,A) = 2d(P,B)$;
   - (b) $d(P,A) + d(P,B) = 10$;
   - (c) $d(P,A) - d(P,B) = 2$;
   - (d) $\overleftrightarrow{PA} \perp \overleftrightarrow{PB}$;
   - (e) slope of $\overleftrightarrow{PA}$ = twice the slope of $\overleftrightarrow{PB}$;
   - (f) slope of $\overleftrightarrow{PA} = 1 +$ slope of $\overleftrightarrow{PB}$;
   - (g) measure of $\angle APB = 45^\circ$;
   - (h) sum of the measures of $\angle A$ and $\angle B$ is 120° ;
   - (i) area of $\triangle ABP = 20$;
   - (j) $d(P,A) < d(P,B)$.

4. The circle whose equation is $x^2 + y^2 = 36$ contains the point $A = (6,0)$. If $P = (x,y)$ is any other point of the circle, find an equation for the locus of the midpoints of $\overline{AP}$.

5. The circle whose equation is $x^2 + y^2 = 25$ contains the point $B = (0,5)$. If $Q = (x,y)$ is any other point of the circle, find an equation for the locus of points P such that Q is the midpoint of $\overline{BP}$.

6. The circle whose equation is $x^2 + y^2 = 100$ contains the point $C = (-10,0)$. A line through C meets the circle again at D , and the line $x = 20$ at E . Find an equation for the locus of the midpoint of $\overline{DE}$, for all positions of the line through C .

7. Find an equation for the locus of the midpoints of all chords of the circle $x^2 + y^2 - 4x + 8y = 0$ which are parallel to the line $y = 3x + 5$.

8. Find an equation for the line containing the midpoints of all chords of the ellipse $x^2 + 9y^2 = 36$ which are parallel to the line $x + y = 10$.

9. Find equations for the families of curves described below:
   (a) All lines which, with the polar axes, form a triangle whose area is 12.
   (b) All lines, the sum of whose intercepts is 6 .
   (c) All circles tangent to the y-axis.
   (d) All circles tangent to the x-axis.
   (e) All circles with radius 1 that are tangent to the line $4x + 3y - 2 = 0$.
   (f) All circles tangent to the line $4x + 3y - 2 = 0$.
   (g) All circles of radius 6 such that the origin is an interior point.
   (h) All circles which go through the origin.
   (i) All circles which go through the point $(12,5)$.
   (j) All circles whose interior contain the origin.
   (k) All circles of radius 5 , such that the origin is not a point of the circle or its interior.
   (ℓ) All circles of radius d which are tangent to the line $ax + by + c = 0$.
   (m) All circles tangent to the lines $3x - 4y + 5 = 0$ and $4x - 3y + 9 = 0$.
   (n) All circles tangent to the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.
   (o) All circles which intersect or touch the x-axis.

(p) All circles which do not intersect or touch the y-axis.

(q) All circles which do not intersect or touch the line $ax + by + c = 0$.

(r) All circles in the interior of $x^2 + y^2 = 100$.

(s) All circles which intersect or touch the circle $x^2 + y^2 = 1$.

(t) All lines which intersect or touch the circle $x^2 + y^2 = 1$.

(u) All circles in the interior of the triangle determined by the points $O = (0,0)$, $A = (10,0)$ and $B = (0,10)$.

(v) All circles whose interiors contain the points A , B , and O of the previous exercise.

(w) All circles which are tangent internally to $x^2 + y^2 = 100$.

(x) All circles which are tangent externally to $x^2 + y^2 = 100$.

(y) All circles to which the circle $x^2 + y^2 = 100$ is tangent internally.

(z) All circles tangent to the line $ax + by + c = 0$ and passing through the point (r,s) .

10. Sketch the graphs of the following conditions.

(a) $|x| = 3$

(b) $|y + 2| = 7$

(c) $|y| < 5$

(d) $|x - 3| \leq 4$

(e) $x^2 + y^2 > 1$

(f) $x^2 < y$

(g) $|x| < |y|$

(h) $|x| + |y| = 6$

(i) $x^2 < x + 20$

(j) $y^2 > 3y$

(k) $xy + 2x > y + 2$

(ℓ) $xy + 3x + 4y > -12$

(m) $5x - 2y + 10 > xy$

(n) $xy = 3y - x + 3$

(o) $3x + 2y - 6 < xy$

(p) $x^3 + xy^2 = 9x$

(q) $x^3y + xy^3 = xy$ ·

(r) $(x - 3)^2 = (y - 5)^2$

(s) $y = \sqrt{x}$

(t) $x = \sqrt{36 - y^2}$

11. Sketch the graphs of the following pairs of parametric equations.

(a) $\begin{cases} x = t + 1 \,, \\ y = t^2 + 2 \,. \end{cases}$

(b) $\begin{cases} x = \frac{1}{t} \,, \\ y = t^2 \,. \end{cases}$

(c) $\begin{cases} x = 2t - 3 \,, \\ y = 3 - 2t \,. \end{cases}$

(d) $\begin{cases} x = t + 1 \,, \\ y = \sin t \,. \end{cases}$

(e) $\begin{cases} x = t^2 \,, \\ y = \cos t^2 \,. \end{cases}$

(f) $\begin{cases} x > t \,, \\ y = 2t \,. \end{cases}$

(g) $\begin{cases} x < t \,, \\ y = t + 1 \,. \end{cases}$

(h) $\begin{cases} x > 2t \,, \\ y = t^2 \,. \end{cases}$

(i) $\begin{cases} x > t \,, \\ y < t \,. \end{cases}$

(j) $\begin{cases} x < t \,, \\ y > t^2 \,. \end{cases}$

12. Sketch and discuss the polar graphs of the following conditions.

(a) $r = \cos 2\theta$

(b) $r = \cos(\theta + 2)$

(c) $r = \sin(\theta - \frac{\pi}{2})$

(d) $r = 2 \sin 3\theta$

(e) $r = 3 \sin 2\theta$

(f) $r = 1 + \sin \theta$

(g) $r = 2 - \cos \theta$

(h) $r = 1 + 2 \sin \theta$

13. Sketch the graphs of $y = x^2$ and $y = x^4$ with respect to the same axes. Generalize.

14. Sketch the graphs of $y = x$, $y = x^3$ and $y = x^5$ with respect to the same axes. Generalize.

15. Sketch the graph of $y = 3 \sin x + 4 \cos x$. What does it remind you of? Note that this equation can also be written in the form

$$y = 5(\tfrac{3}{5} \sin x + \tfrac{4}{5} \cos x) \quad \text{and that} \quad (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2 = 1 \,.$$

Finally, use these facts and a well known trigonometric identity to write a third form of the original equation.

16. Generalize the result of the preceding exercise by considering the equation $y = a \sin x + b \cos x$, where a and b are arbitrary real numbers.

17. Prove analytically that if a set of points in a plane is symmetric with respect to each of two mutually perpendicular lines, it is symmetric with respect to their intersection.

18. Prove that the graph of the pair $x = at + b$, $y = f(t)$ of parametric equations is identical with the graph of the equation $y = f(\frac{x - b}{a})$ obtained by eliminating t in the natural way. Thus there are cases in which it is possible to eliminate a parameter without getting into trouble.

19. Make a graph of $y = a + b \sin (cx + d)$ for each of the following sets of values of a , b , c , d .

    (a) $a = 2$, $b = 3$, $c = 2$, $d = \frac{\pi}{2}$.

    (b) $a = -3$, $b = 2$, $c = -3$, $d = \pi$.

    (c) $a = 3$, $b = -2$, $c = 2$, $d = \frac{3\pi}{2}$.

    (d) $a = -2$, $b = 2$, $c = 3$, $d = 0$.

Challenge Exercises

1. Sketch the rectangular graph of $y = \sin 4x \sin x$. Discuss the graph of $y = (6 + \sin x) \sin 12x$, and generalize suitably. Consider $y = \sin 1000\pi t \cdot \sin 1000000\pi t$, which is related to equations which describe amplitude modulation, in radio broadcasting.

2. For discussion and experiment, if an oscilloscope is available. Adjust the controls to get a stationary sine wave on the screen, then alter one control at a time to change the amplitude, the wave-length, the frequency, etc. If available and possible, find the constants of the oscilloscope and write the actual equations of the curve.

Chapter 7

CONIC SECTIONS

7-1. Introduction

This chapter is intended to give you a better understanding of the curves called conic sections. When you studied geometry, you investigated properties of a circle. In your study of algebra you worked with equations of the various conic sections and their properties. Here we shall first consider briefly the history of conic sections. Then we shall give a formal definition of a conic section and use polar coordinates to obtain a standard polar equation of a conic section. We shall see how equations in polar form are related to the equations in rectangular form that you have already studied. We shall derive properties of these curves and work with some of their many applications.

In studying conic sections you will use the knowledge and techniques acquired so far in analytic geometry. Both rectangular and polar coordinates will be used; often parametric representation will be helpful. Ideas of locus and curve sketching will be used.

It is assumed that you have studied the definitions, equations, and properties of the conic sections; brief summaries will show you what you are expected to know. If you find that you need more detail, you will find it in the following sections of Intermediate Mathematics:

6-3. The Parabola (pages 315-321)
6-4. The General Definition of the Conic (pages 326-331)
6-5. The Circle and the Ellipse (pages 333-336)
6-6. The Hyperbola (pages 342-348)

7-2. History and Applications of the Conic Sections

The curves called conic sections were so named after their historical discovery as intersections of a plane and a surface called a right circular cone. A right circular cone is the surface generated by a line moving about a circle and containing a fixed point on the normal to the plane of the circle at the center of the circle. The fixed point, called the vertex, separates the surface into two parts called nappes. Each line determined by the vertex and a point of the circle is called an element of the cone. The normal to the plane of the circle containing the vertex is called the axis of the cone. The proper conic sections are circles, ellipses, parabolas, and hyperbolas.

The discovery of the conic sections is attributed to the Greek mathematician Menaechmus (circa 375-325 B.C.), who was a tutor to Alexander the Great. He apparently used them in an attempt to solve three famous problems, the trisection of an angle, the duplication of a cube, and the squaring of a circle. Although the Greek mathematicians were primarily interested in the mathematical applications of the conic sections, they did know some of the optical properties of the curves. The definition of the conic sections which we shall use is attributed to Apollonius, who flourished before 200 B.C.

Further discoveries of the physical applications of the conic sections did not occur until the conjectures of the German scientist and mathematician Johannes Kepler (1571 - 1630), who hypothesized that the planets moved in elliptic orbits with the sun as a focus. The theoretical development of Kepler's conjectures followed the gravitation theory and calculus developed by Isaac Newton (1642 - 1727). In fact, it may be shown that any physical object subject to a force which is described by what is called an inverse square law will move in an orbit which is a conic section. Gravity is such a force; the electrical force between charged bodies was found to be another such force by Charles Augustin de Coulomb (1736 - 1806).

Today we find applications of the theory of conic sections in the orbits of planets, comets, and artificial satellites. The theory also applies to the lenses of telescopes, microscopes, and other optical instruments, weather prediction, communication by satellites, geological surveying, and the construction of buildings and bridges. Conics also occur in the study of atomic structure, the long range guidance systems for ships and aircraft, the location of hidden gun emplacements and the detection of approaching enemy ships and aircraft. The surfaces of revolution formed by the conic sections, which will be considered in Chapter 9, find application in the sciences dealing with light, sound, and radio waves.

It is helpful to visualize the four conic sections formed by the intersections of a plane and a right circular cone. We illustrate the physical possibilities below.

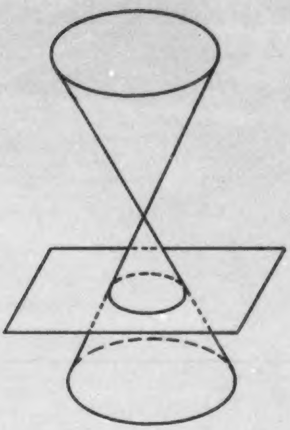


Figure 7-1a: Circle

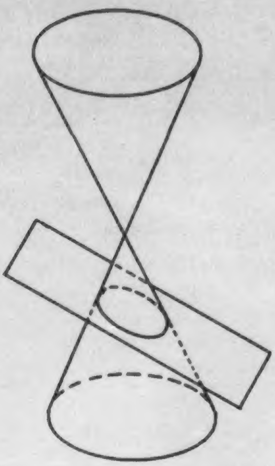


Figure 7-1b: Ellipse

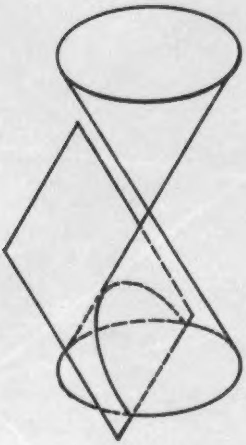


Figure 7-1c: Parabola

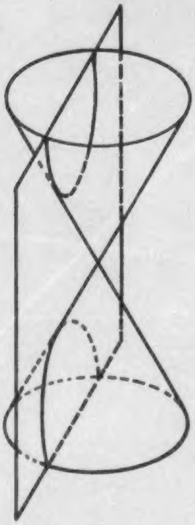


Figure 7-1d: Hyperbola

A circle (Figure 7-1a) is the intersection of a cone and a plane perpendicular to the axis of the cone. An ellipse (Figure 7-1b) is the intersection of a cone and a plane which forms an acute angle with the axis. The measure of this acute angle is greater than the measure of the angle formed by the axis and an element of the cone. A parabola (Figure 7-1c) is the intersection of a cone and a plane parallel to an element of the cone. A hyperbola (Figure 7-1d) is the intersection of a cone and a plane which forms an angle with the axis whose measure is less than the measure of the angle formed by the axis and an element of the cone. These descriptions suggest that circles and ellipses are the sections formed when planes cut every element of the cone; parabolas are formed when planes cut some elements in one nappe of a cone; hyperbolas are formed when planes cut some elements in both nappes of the cone. Although the drawings of Figure 7-1 are limited, cones are infinite in extent; what is illustrated is only part of the parabola or hyperbola.

For a more complete and systematic geometric development of the conic sections, leading to the definition to be given in the following section, see Supplement to Chapter 7.

7-3. The Conic Sections in Polar Form

We shall choose as a defining characteristic of the conic sections that geometric property which leads most readily to their analytic description. This property relates all the conic sections except the circle.

> DEFINITIONS. A conic section is the locus of points in a plane such that for each point the ratio of its distance from a given point F in the plane to its distance from a given line D in the plane is a given constant e. The given point F is called a focus or focal point of the conic section. The given line D is a directrix of the conic section. The given constant e is the eccentricity of the conic section. If $0 < e < 1$, the conic section is called an ellipse. If $e = 1$, the conic section is called a parabola. If $e > 1$, the conic section is called a hyperbola.
>
> A circle is also a conic section and is the locus of points at a given distance from a given point. The given distance is called the radius of the circle and the given point is called the center of the circle.

In some ways it is simpler to describe the conic sections in polar coordinates. We are already familiar with the polar equation, or equation in polar coordinates, of a circle with center at the origin as $r = k$, where k is the radius.

We shall assume that the focal point does not lie on the directrix. Let the focus of the conic section be at the pole and let the directrix be perpendicular to the polar axis. Let the polar axis be oriented away from the directrix; that is, the ray that is the polar axis does not intersect the directrix. Let p be the distance from the pole to the directrix and let $P = (r,\theta)$ be a point of the conic section.

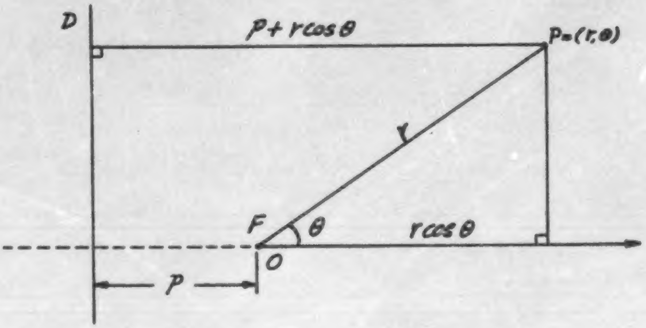


Figure 7-2

Then the distance from P to the focal point is r , and the distance from P to the directrix is $p + r\cos\theta$. Thus,

$$\frac{r}{p + r\cos\theta} = e .$$

Expressing r in terms of θ , we obtain

(1) $$r = \frac{ep}{1 - e\cos\theta} .$$

In the above discussion we have assumed that the focal point did not lie on the directrix. If it does, we obtain certain figures which are called degenerate conics. Geometrically, they are the intersections of cones and planes containing the vertex of the cone. (For a more complete discussion, see Supplement to Chapter 7.)

If the focal point is on the directrix, then $p = 0$, and we may not perform certain algebraic operations, since division by zero would be indicated. We may express the analytic condition as follows:

$$r = er\cos\theta .$$

If $0 < e < 1$, we have $r < r\cos\theta$, which is never true. If $e = 0$, we have $r = 0$, which is an equation of the pole. This is sometimes called a point-circle. (It is sometimes convenient to think of a circle as a special case of the ellipse. This is not consistent with our approach here, but it suggests why one may encounter the description of this locus as a point-ellipse.)

If $p = 0$ and $e = 1$, we obtain $r = r \cos \theta$. From this we may infer either $r = 0$, or $1 = \cos \theta$. The graph of $r = 0$ has just been discussed. The graph of $1 = \cos \theta$ is the line containing the polar axis; this we call a degenerate parabola. If $p = 0$ and $e > 1$, the equation $r = er \cos \theta$ will be satisfied when $\cos \theta = \frac{1}{e}$. Thus the locus is two distinct lines through the pole and is called a degenerate hyperbola. (There will be further discussion of degenerate conics in the Supplement to Chapter 7.)

Thus far we have considered the equation of a conic only in the case in which the focus is at the pole, the directrix is perpendicular to the polar axis, and the polar axis is oriented away from the directrix. Certain other cases will be considered in Example 2 and the exercises, but we shall not take up the case in which the directrix is oblique to the polar axis until we have studied rotation of the axes in Chapter 10.

Example 1. A fixed point F is 4 units from a given line L. Write an equation for the locus of points equidistant from P and L.

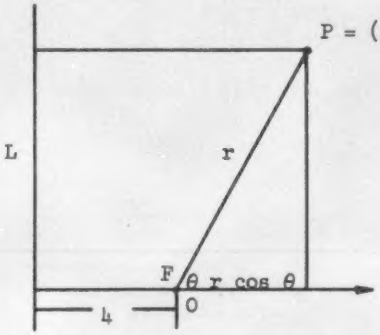
Solution. We place the pole of our polar coordinate system at F, and the polar axis perpendicular to L and directed away from L. Then for any point $P = (r,\theta)$ on the locus,

$$r = 4 + r \cos \theta ,$$

which becomes

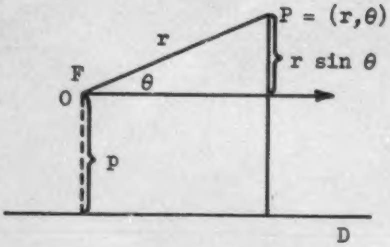
$$r = \frac{4}{1 - \cos \theta} .$$

This equation is in the form of Equation (1), and represents a parabola.



Example 2. What is a polar equation of a conic section with focus at the pole and directrix parallel to the polar axis and p units below it?

Solution. Let $P = (r,\theta)$ be a point of the curve. Then the distance from P to the focal point is r, and the distance from P to the directrix is $p + r \sin\theta$. Thus,



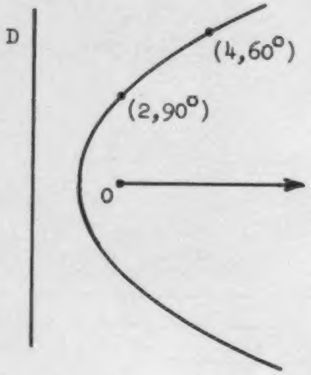
$$\frac{r}{p + r\sin\theta} = e .$$

Expressing r in terms of θ, we obtain

$$r = \frac{ep}{1 - e\sin\theta} .$$

Example 3. Graph $r = \dfrac{2}{1 - \cos\theta}$.

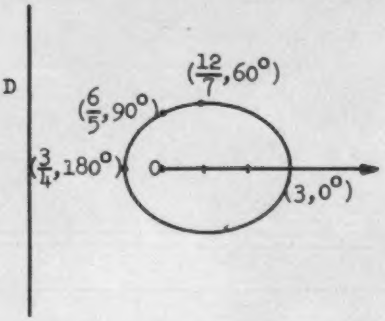
Solution. This equation is in the form of Equation (1) with $e = 1$, $p = 2$. Hence its graph is a parabola with focus at O, and directrix D perpendicular to the polar axis and 2 units to the left of the pole. The vertex must be midway between O and D. Location of one or two more points --say $(4,60^\circ)$ and $(2,90^\circ)$ --and use of symmetry then permit making a sketch.



Example 4. Graph $r = \dfrac{6}{5 - 3\cos\theta}$.

Solution. To obtain the form of Equation (1), we divide numerator and denominator of the fraction by 5, and write the numerator as the product of e and a number which must be p. We obtain

$$r = \frac{\frac{3}{5} \cdot 2}{1 - \frac{3}{5}\cos\theta}.$$



Since $e = \frac{3}{5}$ and $p = 2$, the graph is an ellipse; one vertex divides the normal segment joining the focus to the directrix in the ratio 3 to 5. We obtain a few more points--say $(3,0^\circ)$, $(\frac{12}{7},60^\circ)$, and $(\frac{6}{5},90^\circ)$ --and use symmetry to complete the graph.

Exercises 7-3

Graph each of the following:

1. $r = \frac{4}{1 - \cos\theta}$
2. $r = \frac{6}{2 - 2\cos\theta}$
3. $r = \frac{4}{2 - \cos\theta}$
4. $r = \frac{6}{3 - \cos\theta}$
5. $r = \frac{12}{4 - 5\cos\theta}$
6. $r = \frac{24}{2 - 6\cos\theta}$
7. $r = \frac{4}{1 - \sin\theta}$
8. $r = \frac{6}{2 - 2\sin\theta}$
9. What is a polar equation of a conic section with focus at the pole and directrix parallel to the polar axis and p units above it?
10. What is a polar equation of a conic section with focus at the pole and directrix perpendicular to the polar axis and p units to the right of the pole?
11. Using the results of Exercises 9 and 10, graph the following:

    (a) $r = \frac{4}{1 + \cos\theta}$

    (b) $r = \frac{12}{4 + 5\sin\theta}$

    (c) $r = \frac{8}{4 + 3\sin\theta}$

    (d) $r = \frac{10}{5 + 3\cos\theta}$

In Exercises 12-19, rewrite the equations in a form convenient for graphing, identify the conic section, and sketch the graph.

12. $r - 6 - r\cos\theta = 0$
13. $r - 10 - r\sin\theta = 0$
14. $3r - 12 - 2r\cos\theta = 0$
15. $3r - 12 - 4r\cos\theta = 0$
16. $r = 2 + r\sin\theta$
17. $r = 3 - 2r\cos\theta$
18. $\cos\theta = 1 - \frac{3}{r}$
19. $\sin\theta = \frac{r+2}{r}$
20. An artificial satellite has the center of the earth as its focus. For a polar coordinate system in the plane of its orbit the distance of the satellite from the center of the earth at $\theta = 180^\circ$ is 5000 mi. and at $\theta = 90^\circ$ is 6000 mi. Assuming that the axis is along the line $\theta = 0^\circ$, find the equation describing the orbit and the greatest distance of the satellite from the center of the earth.

7-4. Conic Sections in Rectangular Form

We have developed polar equations for the conic sections in certain specified positions. For a circle with center at the pole, we have

$$r = k .$$

For the other conic sections with focus at the pole, and directrix perpendicular to the polar axis and p units to the left of the pole, we have

$$r = \frac{ep}{1 - e\cos\theta}, \quad \text{representing} \begin{cases} \text{a parabola if } e = 1 , \\ \text{an ellipse if } 0 < e < 1 , \\ \text{a hyperbola if } e > 1 . \end{cases}$$

We shall find the corresponding rectangular equations by using the following equations, developed in Section 2-4:

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x},\ x \neq 0 .$$

Circle: If

$$r = k ,$$

then

$$r^2 = k^2 .$$

(This is equivalent to multiplying the members of $r - k = 0$ by the corresponding members of $r + k = 0$. Since these are both equations of the same circle, the graph of the resulting equation is the same as that of the original equation.)

Since $$r^2 = x^2 + y^2 ,$$

we may write $$x^2 + y^2 = k^2 .$$

We now consider the general equation

$$r = \frac{ep}{1 - e \cos \theta} . \tag{1}$$

We multiply both members of the equation by $1 - e \cos \theta$ to obtain

$$r - er \cos \theta = ep$$

or $$r = e(r \cos \theta + p) ,$$

and square both members of the latter equation to obtain

$$r^2 = e^2(r^2 \cos^2 \theta + 2pr \cos \theta + p^2) . \tag{2}$$

(Whenever we square both members of an equation we must be careful of the interpretation of the result. We have in effect multiplied both members of $r - e(r \cos \theta + p) = 0$ by the corresponding members of $r + e(r \cos \theta + p) = 0$. We recall from Section 5-2 that $r - e(r \cos \theta + p) = 0$ has the related polar equation

$$-r - e\Big((-r) \cos (\theta + \pi) + p\Big) = 0 .$$

Since $\cos (\theta + \pi) = - \cos \theta$, this is equivalent to

$$-r - e(r \cos \theta + p) = 0$$

or $$r + e(r \cos \theta + p) = 0 . \tag{3}$$

Since the "factors" of Equation (2) are equivalent to Equation (1) and its related polar equation, it has the same graph as Equation (1). We may now proceed with the original discussion.) Using $r^2 = x^2 + y^2$ and $r \cos \theta = x$, we have

$$x^2 + y^2 = e^2(x^2 + 2px + p^2) . \tag{4}$$

We now have our equation in rectangular coordinates and wish to examine it for the different values of e.

Parabola: Since $e = 1$, Equation (4) becomes

$$x^2 + y^2 = x^2 + 2px + p^2$$

or

$$y^2 = 2p(x + \frac{p}{2}).$$

This equation, as you may recognize from your study of algebra, represents a parabola with focus at the origin and vertex at $(-\frac{p}{2}, 0)$.

Example 1. Write in rectangular form and sketch the graph of

$$r = \frac{6}{1 - \cos\theta}.$$

Solution. The given equation yields $r - r\cos\theta = 6$, which after transformation becomes

$$\sqrt{x^2 + y^2} - x = 6$$

or

$$\sqrt{x^2 + y^2} = x + 6.$$

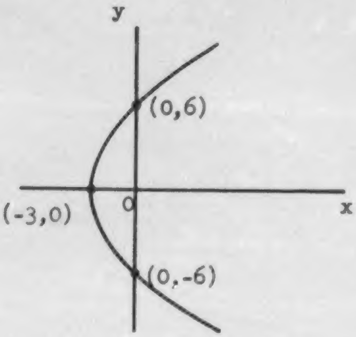
Therefore $x^2 + y^2 = x^2 + 12x + 36$,

and finally

$$y^2 = 12x + 36$$

or

$$y^2 = 12(x + 3).$$



Ellipse: Here $0 < e < 1$. We rewrite Equation (4) as

$$x^2 + y^2 = e^2x^2 + 2e^2px + e^2p^2.$$

We rearrange the terms to obtain

$$(1 - e^2)x^2 - 2e^2px + y^2 = e^2p^2.$$

Since we are looking for a form that we can recognize as the equation of a conic that has a center, we use the technique of completing the square. Dividing by the coefficient of x^2, we have

$$x^2 - 2\frac{e^2p}{1 - e^2}x + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2}$$

or $$x^2 - 2\frac{e^2p}{1 - e^2}x + \left(\frac{e^2p}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2$$

or $$\left(x - \left(\frac{e^2p}{1 - e^2}\right)\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2 .$$

Since $0 < e < 1$. $e^2 < 1$ and $1 - e^2 > 0$. Thus the coefficients of x^2 and y^2 are both positive. Although the equation above is quite cluttered with constants, it should be apparent that it has the form of the equation of an ellipse with center at $\left(\frac{e^2p}{1 - e^2}, 0\right)$.

Example 2. Write in rectangular form:

$$r = \frac{6}{1 - \frac{1}{2}\cos\theta} .$$

Solution. The given equation yields $r - \frac{1}{2}r\cos\theta = 6$ which, by substitution, becomes

$$\sqrt{x^2 + y^2} - \frac{1}{2}x = 6 .$$

Therefore $$\sqrt{x^2 + y^2} = \frac{1}{2}x + 6 ;$$

hence $$x^2 + y^2 = \frac{1}{4}x^2 + 6x + 36 .$$

Finally, this becomes $3x^2 + 4y^2 - 24x - 144 = 0$, which you may recognize as an equation for an ellipse in rectangular form. We may write this in standard form thus:

$$3(x^2 - 8x + 16) + 4y^2 = 144 + 48 ,$$

or $$3(x - 4)^2 + 4y^2 = 192 ,$$

or $$\frac{(x - 4)^2}{64} + \frac{y^2}{48} = 1 .$$

You may recognize that this equation represents an ellipse with center at (4,0) .

Hyperbola: The algebraic manipulation involved in expressing the equation of a hyperbola in rectangular form is identical with that for the ellipse. However, when we reach the form

$$\left(x - \left(\frac{e^2p}{1 - e^2}\right)\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2 ,$$

we note that since $e > 1$, $e^2 > 1$ and $1 - e^2 < 0$. Thus the coefficients of x^2 and y^2 have opposite signs.

It should be apparent that this is the equation of a hyperbola with center at $\left(\frac{e^2p}{1 - e^2}, 0\right)$.

Exercises 7-4

For each of the polar equations below you are asked to do three things:

(a) Sketch the graph.

(b) Write a corresponding equation in rectangular coordinates.

(c) Write the related polar equation.

1. $r = 3$
2. $r = 9$
3. $r = 2 \cos \theta$
4. $r = \cos \theta + \sin \theta$
5. $r = \frac{4}{1 - \cos \theta}$
6. $r = \frac{3}{1 + \cos \theta}$
7. $r = \frac{3}{1 - 2 \cos \theta}$
8. $r = \frac{6}{2 - \cos \theta}$
9. $r = \frac{5}{3 - 2 \cos \theta}$
10. $r = \frac{5}{2 - 3 \cos \theta}$
11. $r = 1 + \cos \theta$
12. $r - r \sin \theta = 2$
13. $4r - 3r \cos \theta = 12$
14. $4r + 5r \sin \theta = 20$

7-5. The Parabola

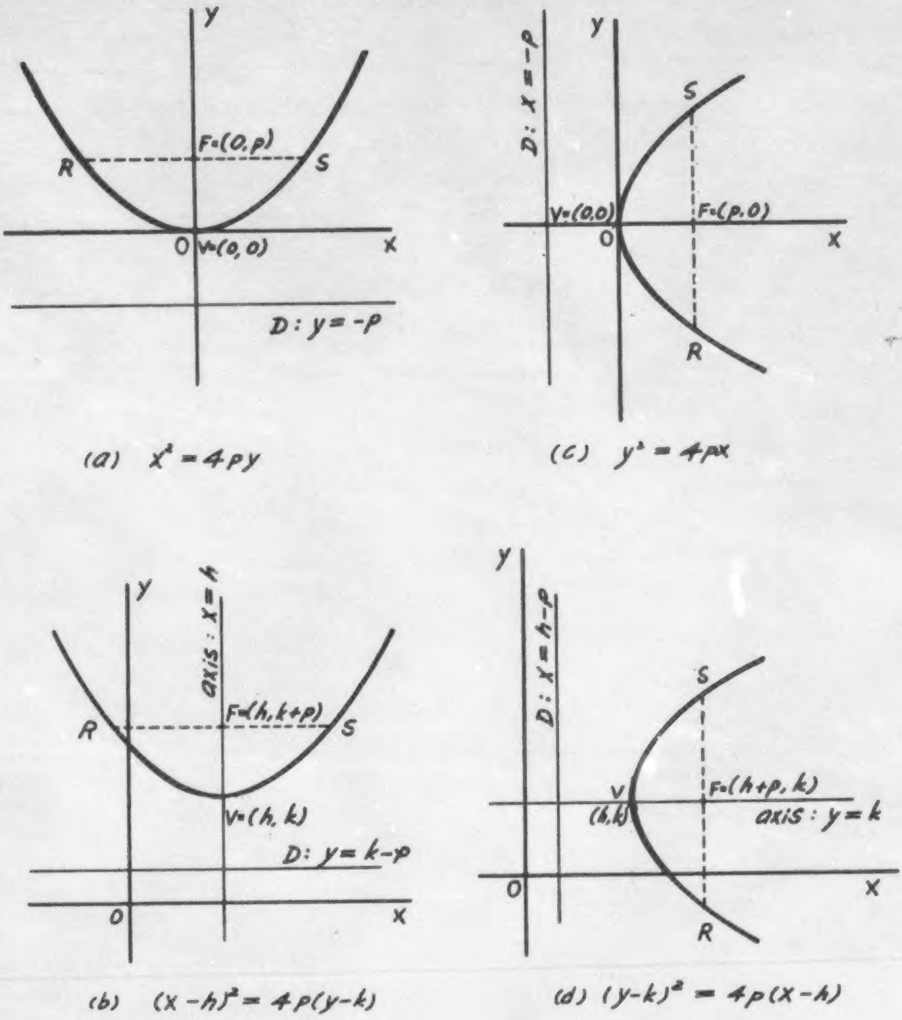
In this section and the following three we consider the four main kinds of conic sections: parabola, circle, ellipse, and hyperbola. There are brief summaries of the important definitions and properties. Equations in rectangular coordinates--often called standard forms--are given for these curves with axes on or parallel to the coordinate axes. Much of this information is not new; it is placed here because of its importance, and for your convenience.

The parabola is defined as the set of points equidistant from a fixed point (the focus) and a fixed line (the directrix). A parabola is symmetric with respect to the line through the focus perpendicular to the directrix. This line of symmetry is called the axis of the parabola, and its point of intersection with the parabola is called the vertex of the parabola.

In Figure 7-3 F, V, and D indicate the focus, vertex, and directrix, respectively, and $|p|$ is the distance between F and V. If $p > 0$, the parabola extends upward or to the right as shown; if $p < 0$, it extends downward or to the left.

In making a quick sketch of a parabola, it is convenient, after locating V, F, and D, to find the length of the latus rectum. This is the chord of the parabola through the focus perpendicular to the axis. If in Equation (a) Figure 7-3 we set $y = p$, we find $x = \pm 2p$; thus, the length of the latus rectum is $|4p|$. (The student should verify that for each of the other standard forms of the equation given in Figure 7-3 the length of the latus rectum is also $|4p|$.)

In general a conic section has been defined as the set of points P such that the ratio of the distance from P to a fixed point, to the distance from P to a fixed line, is a constant e, called the eccentricity. For the parabola $e = 1$.


y
F=(0,p)
R
S
O
V=(0,0)
x
D: y=-p
y
D: x=-p
S
V=(0,0)
F=(p,0)
O
x
R
y
axis: x=h
F=(h,k+p)
R
S
V=(h,k)
D: y=k-p
O
x
y
D: x=h-p
S
V
F=(h+p,k)
(h,k)
axis: y=k
O
x
R

(a) $x^2 = 4py$

(c) $y^2 = 4px$

(b) $(x-h)^2 = 4p(y-k)$

(d) $(y-k)^2 = 4p(x-h)$

Figure 7-3

Our definition of the parabola makes no restriction on the position of the fixed point and line. What if the point is on the line? Our knowledge of geometry tells us that the locus must be the line perpendicular to the directrix at the fixed point. If we let $p = 0$ in, say, Equation (d) of Figure 7-3, we obtain

$$(y - k)^2 = 0 .$$

This equation represents a straight line. This locus is often called a degenerate parabola.

The parabola has important geometric properties, some of which concern tangents; these you will be able to derive more easily when you have studied calculus. One of the best known is the reflective property: light rays parallel to the axis of a parabolic reflector are concentrated at the focus, and light rays emanating from the focus are reflected parallel to the axis. This property, although usually illustrated in two dimensions, has more interest and physical applications in three dimensions. Such parabolic reflectors are used not only for light rays, but also for heat, sound, and microwaves. You may have seen such reflectors used with microphones, or radar antenna, or as parts of artificial satellites.

The parabola is also important in analyzing trajectories; the path of a projectile can be approximated by a parabola. Under certain conditions of loading, the cable of a suspension bridge hangs in the form of a parabola . Arches of bridges sometimes have parabolic form.

Example 1. Rewrite the equation $x^2 + 4x + 8y - 4 = 0$ in standard form. Write the coordinates of the vertex and focus and the equations of the axis and directrix.

Solution. Since x^2 is the only second-degree term, we group the x-terms and complete the square.

$$x^2 + 4x = -8y + 4$$

is equivalent to

$$x^2 + 4x + 4 = -8y + 8 ,$$

or

$$(x + 2)^2 = -8(y - 1) .$$

This last form we may compare with $(x - h)^2 = 4p(y - k)$, and recognize as an equation of the parabola with axis parallel to the y-axis, and vertex $(-2,1)$. Since $p = -2$, the parabola opens downward. The axis is a vertical line through the vertex; hence its equation is $x = -2$. The directrix is a horizontal line 2 units above the vertex and has the equation, $y = 3$. The focus, $(-2,-1)$, is two units below the vertex.

Example 2. Write an equation of the parabola with vertex $(3,2)$ and directrix $x = -1$.

Solution. Since the directrix is vertical, the axis is horizontal; an equation will be in the form (d) of Figure 7-3. The distance from V to the directrix is p ; here $p = 4$. Thus an equation is

$$(y - 2)^2 = 16(x - 3) .$$

Exercises 7-5

1. Rewrite each of the following equations in standard form; write the coordinates of vertex and focus, and equations of axis and directrix; draw the graph.

   (a) $x^2 = -16y$

   (b) $y^2 = 16x$

   (c) $5x^2 - 3y = 0$

   (d) $y^2 - 5y + 6x - 16 = 0$

   (e) $2x^2 - 8x - 3y + 11 = 0$

   (f) $y = ax^2 + bx + c$

2. We have noted that a special or degenerate case of the parabola occurs when the fixed point is on the fixed line. In this case Equation (d) of Figure 7-3 becomes $(y - k)^2 = 0$; the locus is a straight line parallel to the x-axis.

   (a) Find the degenerate case of each of the other standard forms of the equation of the parabola, and state what the locus is.

   (b) If a parabola is a section of a cone by a plane parallel to an element of the cone, can you explain these "degenerate parabolas" as limiting cases?

3. Derive an equation of a parabola to fit each of the following conditions by using the locus definition of a parabola.

   (a) Focus $(-1,-2)$, directrix $x = 2$

   (b) Focus $(-1,3)$, directrix $y = 2$

   (c) Vertex $(0,0)$, focus $(-5,0)$

   (d) Vertex $(4,5)$, directrix $x = 3$

4. Obtain an equation for each of the parabolas for which conditions are given in Exercise 3 by using the standard forms of the equations.

5. Find an equation of a parabola to fit each of the following conditions.

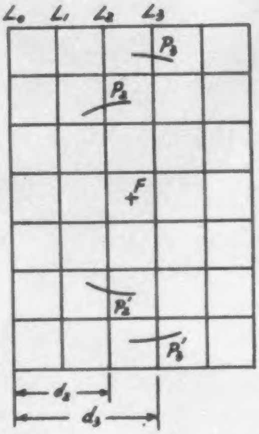
   (a) Vertex $(0,0)$, directrix $2x - 5 = 0$

   (b) Vertex $(2,-3)$, directrix the x-axis

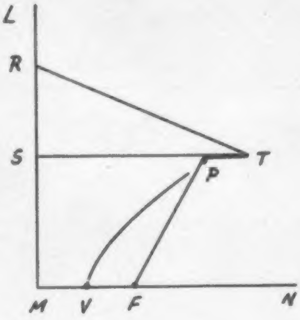
   (c) Vertex $(0,0)$, axis of symmetry the x-axis, passing through the point $(2,7)$

   (d) Latus rectum 16, open down, vertex $(-2,3)$

6. Cross-section paper and a compass can be used as follows. Mark one of the printed lines L_0 and mark successive parallel lines L_1, L_2, ... Select any point F on the same side of L_0 as L_1. With a compass measure on one of the printed lines the distance d_2 from L_0 to L_2. With d_2 as radius and F as center, locate points P_2 and P_2' on L_2. In a similar fashion, using d_3 as radius, locate P_3 and P_3' on L_3. Prove that the points P_2, P_2', ... lie on a parabola.



7. To construct a parabola mechanically, place a straight edge L perpendicular to the line MN. Attach one end of a piece of string of length ST to point T of right triangle RST, and the other end to a point F on MN. With a pencil, hold the string against the side ST of the triangle as the side SR slides along ML. Prove that the point P of the pencil describes a parabola as the triangle slides.



Challenge Problems

1. In Section 6-7 it was shown that the family of tangents to the parabola $y = x^2$ at any point $P = (a,a^2)$ on the parabola can be represented by the equation $y = 2ax - a^2$. Prove the reflective property of the parabola for this case. (Hint: Show that the tangent makes equal angles with the line from P to the focus and the line through P parallel to the axis of the parabola.)

2. Again using the results of Section 6-7, prove the following statements for the parabola $y = x^2$.
   (a) The points of tangency of two perpendicular tangents are collinear with the focus.
   (b) The locus of the intersections of pairs of perpendicular tangents is the directrix.

7-6. The Circle

A circle is the set of points in a plane each of which is at a given distance from a fixed point of the plane. If the fixed point, called the center, is $C = (h,k)$, and the given distance is r, for the required set of points $P = (x,y)$ we have

$$(x - h)^2 + (y - k)^2 = r^2 .$$

If $r = 0$, the solution set is the single point (h,k); such a locus is often called a point-circle. If $r^2 < 0$, the solution set is the empty set; in this case the locus is sometimes said to be an imaginary circle.

Since there are three arbitrary constants h, k, r in the standard equation of a circle, it is in general possible to impose three geometric conditions on a circle. The following example will illustrate this.

Example 1. Find an equation of the circle which passes through the three points $(1,2)$, $(-1,1)$, $(2,-3)$.

Solution A. Using the equation $x^2 + y^2 + Dx + Ey + F = 0$, we write in turn the condition that each of the given points satisfies the equation.

$$1 + 4 + D + 2E + F = 0\ ,\ \text{or}\ \ D + 2E + F = -5$$

$$1 + 1 - D + E + F = 0\ ,\ \text{or}\ -D + E + F = -2$$

$$4 + 9 + 2D - 3E + F = 0\ ,\ \text{or}\ 2D - 3E + F = -13$$

We now have a system of 3 equations in 3 unknowns; solving these by any desired method, we find that

$$D = -\frac{23}{11}\ , \qquad E = \frac{13}{11}\ ,\ \text{and} \qquad F = -\frac{58}{11}\ .$$

We substitute these values in the equation and multiply by 11 to obtain

$$11x^2 + 11y^2 - 23x + 13y - 58 = 0\ .$$

Solution B. Here we use the condition that the center (h,k) is equidistant from any two points of the circle. We select the first two points and write this condition.

$$(h - 1)^2 + (k - 2)^2 = (h + 1)^2 + (k - 1)^2\ ,\ \text{or}\ \ 4h + 2k = 3$$

We then do the same thing for the last two points.

$$(h + 1)^2 + (k - 1)^2 = (h - 2)^2 + (k + 3)^2\ ,\ \text{or}\ \ 6h - 8k = 11$$

The coordinates of the center of the desired circle must satisfy both of these equations; solving them, we have

$$C = (h,k) = \left(\frac{23}{22}, -\frac{13}{22}\right)\ .$$

Now we find the radius r, the distance between C and any of the given points, say the first.

$$r^2 = \left(1 - \frac{23}{22}\right)^2 + \left(2 + \frac{13}{22}\right)^2$$

$$= \frac{(-1)^2 + (57)^2}{22^2}$$

$$r = \frac{1}{22}\sqrt{3250}$$

Thus the equation of the circle is

$$(x - \frac{23}{22})^2 + (y + \frac{13}{22})^2 = \frac{3250}{22^2} .$$

The student should satisfy himself that this equation, when simplified, is the same as the one obtained in Solution A. What happens to the solution of this problem if the three points are collinear?

Example 2. What is the locus of $36x^2 + 36y^2 - 36x + 48y + 24 = 0$?

Solution. We regroup the terms and apply the distributive law to obtain

$$36(x^2 - x) + 36(y^2 + \frac{4}{3}y) = -24 .$$

We complete the squares by adding the same numbers to each member of the equation, obtaining

$$36(x^2 - x + \frac{1}{4}) + 36(y^2 + \frac{4}{3}y + \frac{4}{9}) = -24 + 9 + 16 ,$$

which is equivalent to

$$(x - \frac{1}{2})^2 + (y + \frac{2}{3})^2 = \frac{1}{36} .$$

Hence the locus is a circle with center $(\frac{1}{2}, -\frac{2}{3})$, and radius $\frac{1}{6}$.

Exercises 7-6

1. Rewrite the following equations to show what each locus is; if it is a circle, find the center and radius.

   (a) $x^2 + y^2 - 8x = 0$

   (b) $x^2 + y^2 - 6x - 10y + 33 = 0$

   (c) $x^2 + y^2 - 4x + 8y + 20 = 0$

   (d) $x^2 + y^2 + 14x - 9y + 60 = 0$

   (e) $x^2 + y^2 - x + y = 0$

   (f) $x^2 + y^2 - 2ax - 2by + a^2 + b^2 = 0$

   (g) $5x^2 + 5y^2 - 6x + 4y + 2 = 0$

   (h) $2x^2 + 2y^2 - 2ax + 2by - ab = 0$

2. In each of the following, find an equation of the circle (or of each circle) determined by the given conditions and make a sketch. (Let C and r represent center and radius.)
   (a) $C = (3,-5)$, $r = 7$
   (b) $C = (-5,12)$ and passing through the origin
   (c) $C = (3,2)$ and tangent to an axis
   (d) $r = 3$ and passing through the points $(-1,1)$, $(2,4)$,
   (e) $C = (1,2)$ and tangent to the line $3x - 4y - 12 = 0$
   (f) passing through the points $(2,3)$, $(5,1)$, $(0,1)$

3. (a) Use the fact that a tangent to a circle is perpendicular to the radius at the point of contact to find an equation of a tangent to the circle $x^2 + y^2 = 25$ at the point $(3,-4)$.
   (b) Prove that an equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1,y_1) of the circle is $x_1x + y_1y = r^2$.

4. (a) Find the length of a tangent from $(3,7)$ to the circle $x^2 + y^2 = 25$.
   (b) Show that if t is the length of a tangent from the point (x_1,y_1) to the circle $x^2 + y^2 + Dx + Ey + F = 0$,

$$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F .$$

   (c) If in using this formula you find that $t^2 = 0$, how do you interpret this geometrically? What if $t^2 < 0$?

5. In Section 6-5 we considered the family of circles through the common points of two circles; such a family is sometimes called a coaxial family or a pencil of circles.
   (a) Find an equation of a pencil of circles through the intersections of the circles with equations

$$x^2 + y^2 - 10x - 2y - 35 = 0 \text{ and}$$
$$x^2 + y^2 + 4x - 6y - 49 = 0 .$$

   (b) Find an equation of a circle of this pencil which passes through the point $(0,-6)$.
   (c) Find an equation of a circle of this pencil which has its center on the line $x + 5 = 0$.

6. In Section 6-5 we found the equation of a line through the common points of two circles; the same algebraic technique gives us the equation of a line, whether the circles intersect or not. This line is called the radical axis of the two circles. Prove that the tangents drawn to two circles from any point in their radical axis are equal in length.

7. Find the coordinates of a point from which equal tangents can be drawn to the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 - 6x + y = 12$, $x^2 + y^2 + 4x - 3y = 15$.

8. Prove that the radical axis of two circles is perpendicular to the line of centers of the circles.

9. Two intersecting circles are said to be orthogonal if the tangents at each point of intersection are perpendicular. Prove that if circles $x^2 + y^2 + D_1x + E_1y + F_1 = 0$ and $x^2 + y^2 + D_2x + E_2y + F_2 = 0$ are orthogonal, then $D_1D_2 + E_1E_2 = 2(F_1 + F_2)$.

10. Show that the following pairs of circles are orthogonal.

    (a) $x^2 + y^2 + 3x - 5y + 6 = 0$, $x^2 + y^2 + 10x + 9 = 0$

    (b) $2x^2 + 2y^2 + 2x + 1 = 0$, $2x^2 + 2y^2 - 4x + 6y - 3 = 0$

11. Determine the constant k so that each of the following pairs of circles is orthogonal.

    (a) $x^2 + y^2 - 3x + 4y - 3 = 0$, $x^2 + y^2 + 2x - y + k = 0$

    (b) $3x^2 + 3y^2 + kx - 2y = 4$, $5x^2 + 5y^2 - x + 2y = 2$

Challenge Problems

1. The vertices of triangle ABC are the centers of any three circles which intersect each other. Prove that their common chords are concurrent.

2. The vertices of triangle ABC are the centers of any three circles. Prove that their radical axes are concurrent. (Does your proof also hold for Challenge Problem 1 ?)

7-7. The Ellipse

The ellipse is defined as the set of points P such that the distance from P to a fixed point (the focus) is equal to the product of a constant e and the distance from P to a fixed line (the directrix). The constant e, the eccentricity, is such that $0 < e < 1$. In our earlier study we found that if we take as focus $F = (c,0)$, and as directrix the line $x = \frac{c}{e^2}$, and let $a = \frac{c}{e}$ and $b = \frac{c}{e}\sqrt{1 - e^2}$, an equation for the ellipse can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 . \tag{1}$$

We note from these relations that the equation of the directrix can also be written $x = \frac{a^2}{c}$, or $x = \frac{a}{e}$. Another useful relation is $c^2 = a^2e^2 = a^2 - b^2$.

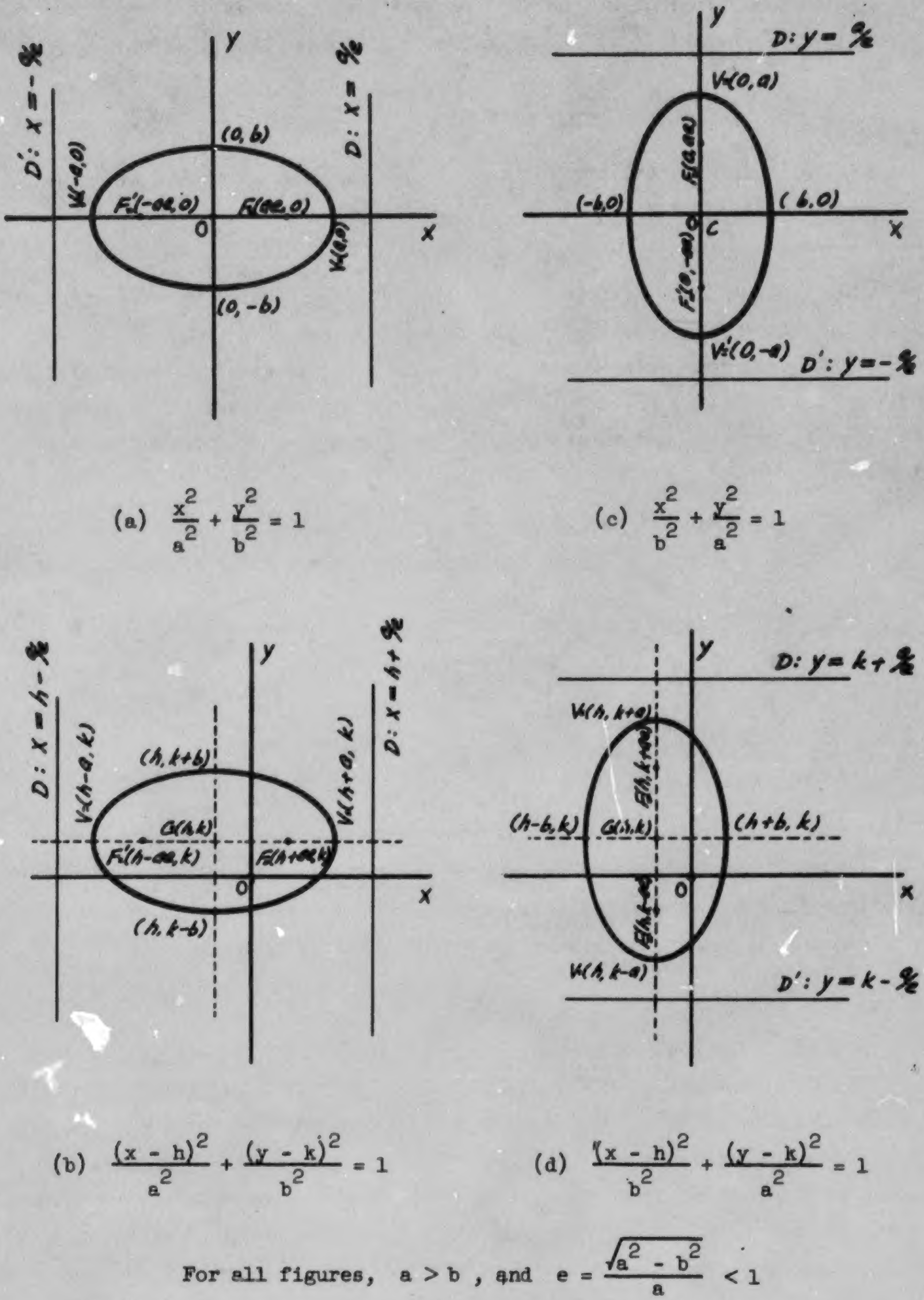
From Equation (1) we see that the graph of the ellipse is symmetric with respect to the origin and to both of the coordinate axes; hence the point $F' = (-c,0)$ and the line $x = -\frac{c}{e^2}$ also serve as focus and directrix. The chord of the ellipse which contains the foci is called the major axis; its endpoints are called vertices. The midpoint of the major axis is called the center of the ellipse; the chord perpendicular to the major axis at the center is called the minor axis.

In Figure 7-4, parts (a) and (c) summarize information about the ellipse with Equation (1), and also the comparable case with the role of the x- and y-axes interchanged.

The equation

$$\frac{(x - h)^2}{M} + \frac{(y - k)^2}{N} = 1 , \tag{2}$$

if M and N are positive, is in the form of an equation of an ellipse with center $C = (h,k)$. Whether the major axis is parallel to the x- or the y-axis depends on whether M or N is larger. Using V,V', F,F', and D,D' to indicate vertices, foci, and directrices, we can summarize in Figure 7-4, parts (b) and (d), information about an ellipse with center (h,k) and axes parallel to the coordinate axes.


D': x = -a/e
V'(-a,0)
F'(-ae,0)
F(ae,0)
V(a,0)
(0,b)
(0,-b)
D: x = a/e
y
x
0
D: y = a/e
V(0,a)
F(0,ae)
(-b,0)
(b,0)
C
F'(0,-ae)
V'(0,-a)
D': y = -a/e
y
X
0
D: x = h - a/e
V'(h-a,k)
(h,k+b)
C(h,k)
F'(h-ae,k)
F(h+ae,k)
V(h+a,k)
D: x = h + a/e
(h,k-b)
y
x
0
D: y = k + a/e
V(h,k+a)
F(h,k+ae)
(h-b,k)
C(h,k)
(h+b,k)
F'(h,k-ae)
V'(h,k-a)
D': y = k - a/e
y
x
0

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(c) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

(b) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

(d) $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

For all figures, $a > b$, and $e = \frac{\sqrt{a^2 - b^2}}{a} < 1$

Figure 7-4

If in Equation (2) M and N are negative, there is no locus; sometimes in this case we speak of an imaginary ellipse. The equation

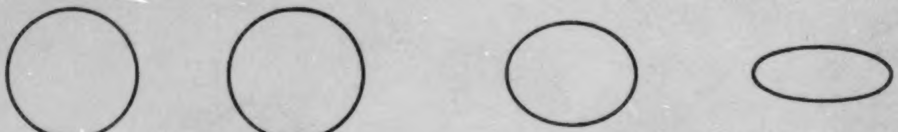
$$\frac{(x-h)^2}{M} + \frac{(y-k)^2}{N} = 0$$

has as its locus only the point (h,k) . Such a locus is spoken of as a degenerate ellipse or a point-ellipse, since its equation resembles that of an ellipse.

In discussing the ellipse and its properties and graph we have, in this section, written the equations in rectangular coordinates. All of the work could have been done using polar coordinates. If the equation of an ellipse, or any conic section, is in polar coordinates, you may leave it in that form in order to graph it and obtain such information as coordinates of foci and vertices.

The shape of an ellipse varies with its eccentricity. As you see in Figure 7-5, the nearer e is to zero, the closer the shape of the ellipse is to a circle. You can see why the circle is spoken of as an ellipse of eccentricity zero. For increasingly large values of e , the ellipse is more and more elongated. Can you explain this result from the fact that

$b = \frac{c}{e}\sqrt{1 - e^2} = a\sqrt{1 - e^2}$?



$e = .1$ $e = .4$ $e = .6$ $e = .9$

Figure 7-5

Perhaps best known of the properties of an ellipse is that, for any point on an ellipse, the sum of the distances to the foci is a constant equal to the length of the major axis. The reflective property has important applications in optics and radar: Since a tangent at any point of an ellipse makes equal angles with the radii drawn to the two foci, rays are reflected from one focus to the other. This property explains the "whispering gallery" effect in some halls, where a whisper at one spot, though not audible nearby, is easily heard at some more remote spot. The orbits of planets and the paths of electrons about the nucleus in an atom are approximately ellipses with the sun and the nucleus respectively at one focus. The elliptic form also occurs in arches and gears.

Example 1. Discuss and sketch the ellipse with equation

$$9x^2 + 4y^2 + 54x - 16y + 61 = 0 .$$

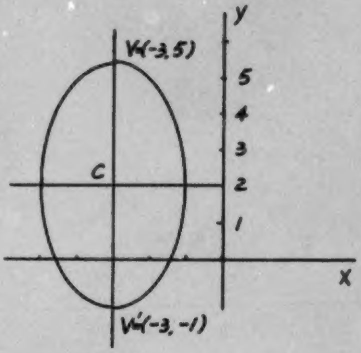
Solution. We proceed to rewrite this equation.

$$9(x^2 + 6x + 9) + 4(y^2 - 4y + 4) = 81 + 16 - 61$$

is equivalent to $$9(x + 3)^2 + 4(y - 2)^2 = 36$$

or $$\frac{(x + 3)^2}{2^2} + \frac{(y - 2)^2}{3^2} = 1 .$$

Since 3 is larger than 2, we see that $a = 3$, $b = 2$, and the major axis is parallel to the y-axis. The curve is an ellipse such as (d) of Figure 7-4 with center $(-3,2)$. The eccentricity



$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{5}}{3}$; hence $ae = \sqrt{5}$ and $\frac{a}{e} = \frac{9}{5}\sqrt{5}$. We use these values and the formulas of Figure 7-4 (d) to obtain the coordinates for the vertices, $V = (-3,5)$, $V' = (-3,-1)$, and foci, $F = (-3,2 + \sqrt{5})$, $F' = (-3,2 - \sqrt{5})$, equations of the axes ($x = -3$, $y = 2$) and directrices ($y = 2 \pm \frac{9}{5}\sqrt{5}$). In making a sketch we usually locate the center first, and mark off from it the semi-axes; the values used for this (h,k,a,b) may all be obtained directly from the equation in form (d) of Figure 7-4.

Example 2. Write an equation of the ellipse with foci $F = (2,4)$ and $F' = (-4,4)$ and with $e = \frac{3}{5}$.

Solution. Since for this ellipse the major axis is parallel to the x-axis, we shall use form (b) of Figure 7-4. The distance between the foci is

$$2ae = |2 - (-4)| = 6 ;$$

therefore $$ae = 3 .$$

Since $e = \frac{3}{5}$, $$a = \frac{ae}{e} = \frac{3}{\frac{3}{5}} = 5 .$$

Using the relation $a^2e^2 = a^2 - b^2$,

we have $b^2 = a^2 - a^2e^2$,

$b^2 = 25 - 9$,

and $b^2 = 16$.

Thus $b = 4$.

Since the center is the midpoint of $\overline{FF'}$, $C = (-1,4)$. We now write the equation

$$\frac{(x + 1)^2}{25} + \frac{(y - 4)^2}{16} = 1 .$$

Exercises 7-7

1. Write an equation of the ellipse with center $(3,2)$, major axis equal to 12 and parallel to the x-axis, and minor axis 8 . Find the eccentricity, the coordinates of the foci and vertices, and the equations of the directrices. Make a sketch.

2. Write an equation of the ellipse with center at $(0,0)$, one vertex $(3,0)$, and one focus $(2,0)$.

3. Rewrite the following equations in the forms of Figure 7-4. For each, find the eccentricity, the coordinates of foci and vertices, and equations of directrices; make a sketch.

   (a) $4x^2 + y^2 = 4$

   (b) $4x^2 + 25y^2 = 100$

   (c) $3x^2 + 2y^2 = 6$

   (d) $4x^2 + 9y^2 = 1$

   (e) $36(x - 4)^2 + 25(y + 3)^2 = 900$

   (f) $4(x + 5)^2 + 9(y + 1)^2 = 36$

   (g) $9x^2 + 4y^2 - 36x = 0$

   (h) $4x^2 + y^2 + 8x - 10y + 13 = 0$

   (i) $16x^2 + 25y^2 - 32x + 150y + 241 = 0$

4. Write an equation of an ellipse to fit each of the following conditions (letters are used as in Figure 7-4).

   (a) $C = (0,0)$; major axis, 8 , parallel to x-axis; minor axis, 6

   (b) $C = (0,0)$; $V = (0,3)$; $F = (0,2)$

   (c) $C = (3,5)$, directrix $x = 10$, $a = 5$

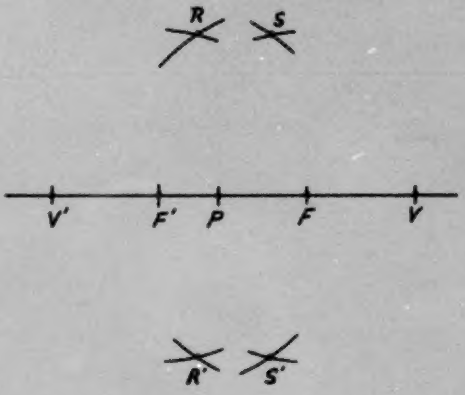
   (d) $F = (3,4)$, $F' = (-1,4)$, $e = \frac{1}{2}$

5. What change must be made in the definition of latus rectum given for the parabola to make it apply to the ellipse? Find a formula for the length of the latus rectum for an ellipse; check that your formula applies for all four cases in Figure 7-4.

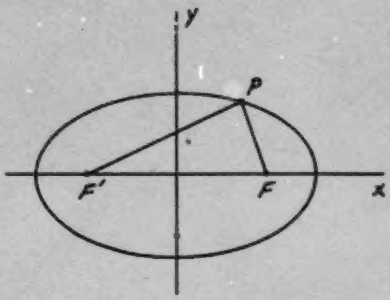
6. A focal radius of an ellipse is a segment drawn from a focus to any point of the ellipse. Prove that the sum of the lengths of the focal radii for any point on an ellipse is a constant, and equal to the length of the major axis.

7. Prove that an ellipse is the locus of points the sum of whose distances from two fixed points is a constant greater than the distance between the two fixed points.

8. Construct some points of an ellipse from given vertices V, V' and foci F, F' as follows. Select any point P of the segment $\overline{V'V}$. With F as center and PV as radius, strike arcs above and below $\overline{V'V}$. With F' as center and PV' as radius, describe arcs intersecting the ones first drawn, and locating points R and R' of the ellipse. Then interchange F and F' and repeat, locating two more points, S and S' . Thus for any point such as P on the segment four points can be located. Why do the points so located lie on the ellipse with the given foci and vertices?



9. Construct an ellipse as follows. Tie the ends of a piece of string to two thumbtacks. Stick the tacks into a piece of cardboard at F and F' . Draw the string taut with a pencil point (P) and trace a curve. Why is the curve an ellipse? Keeping the length of the string the same, change the distance between the tacks and repeat the construction. What do you observe?



10. Use the locus definition in Exercise 7 in deriving equations of
    (a) an ellipse with fixed points $(2,3)$ and $(6,3)$ and sum of focal radii equal to 6.
    (b) an ellipse with fixed points $(1,1)$ and $(3,5)$ and sum of focal radii equal to 6.

11. Some writers like to include the circle as a special case of an ellipse. If a circle with its center at the origin is to be thought of as an ellipse, then $a = b$. What, then, is e ? Is this consistent with the focus-directrix definition of a conic?

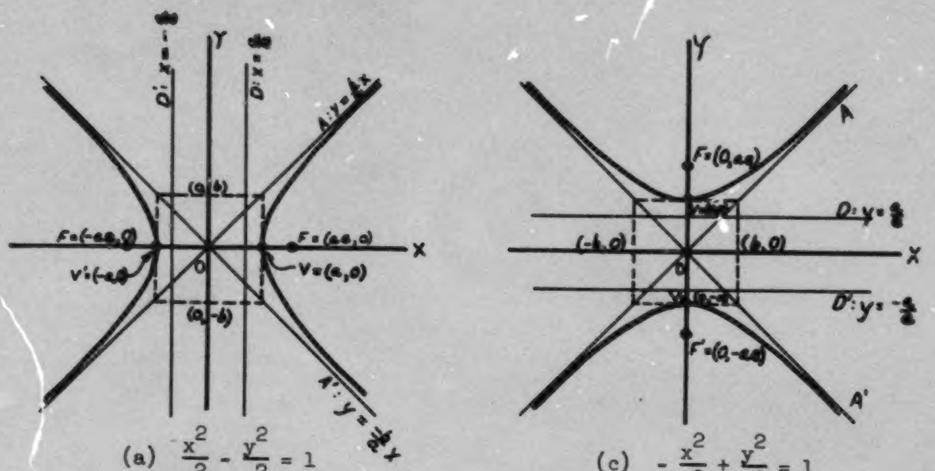
12. Show that the ellipse with focus $F = (c,0)$, eccentricity e, and directrix $x = \frac{c}{e^2}$ has another focus $F' = (-c,0)$ and another directrix $x = -\frac{c}{e^2}$.

13. Discuss and sketch the graph of $r = \frac{6}{2 - \cos\theta}$, including coordinates of the vertices, foci, and center; the lengths of the major and minor axes and of the latus rectum; eccentricity.

14. Prove that in an ellipse the length of the major axis is the mean proportional between the distance between the foci and the distance between the directrices.

7-8. The Hyperbola

The hyperbola is defined as the set of points P such that the distance from P to a fixed point (the focus) is the product of the eccentricity, e , and the distance from P to a fixed line (the directrix), with e greater than one. In our earlier study we found that if, as with the ellipse, we take as focus $F = (c,0)$ and as directrix the line $x = \frac{c}{e^2}$, and let $a = \frac{c}{e}$ and

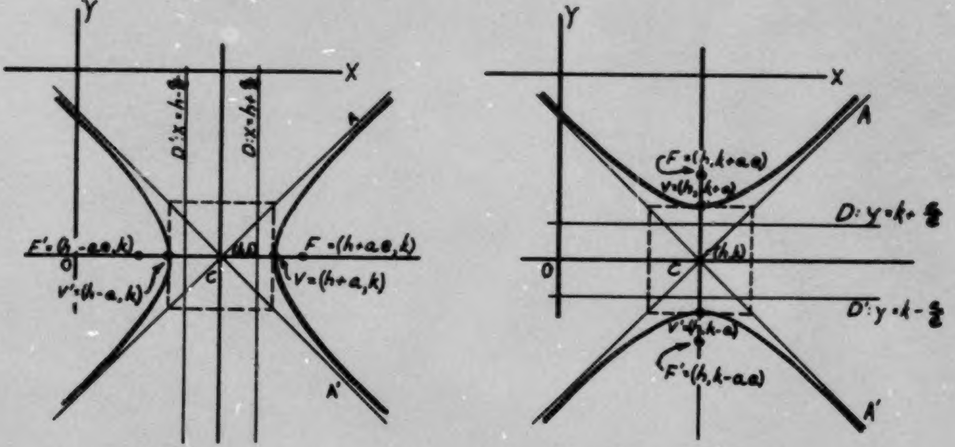

D': x = -a/e
Y
D: x = a/e
A: y = b/a x
(a,b)
F=(-ae,0)
F=(ae,0)
X
V'(-a,0)
O
V=(a,0)
(a,-b)
A': y = -b/a x
Y
A
F=(0,ae)
V=(0,a)
D: y = a/e
(-b,0)
(b,0)
X
O
D': y = -a/e
V'=(0,-a)
F'=(0,-ae)
A'

(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$)

(c) $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

(Asymptotes : $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 0$)


Y
X
D': x = h - a/e
D: x = h + a/e
A
F'=(h-ae,k)
F=(h+ae,k)
(h,k)
O
C
V'=(h-a,k)
V=(h+a,k)
A'
Y
X
A
F=(h,k+ae)
V=(h,k+a)
D: y = k + a/e
(h,k)
O
C
D': y = k - a/e
V'=(h,k-a)
F'=(h,k-ae)
A'

(b) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$

(d) $-\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

(Asymptotes: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 0$) (Asymptotes : $-\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 0$)

For all figures, $e = \frac{\sqrt{a^2 + b^2}}{a} > 1$

Figure 7-6

$b = \frac{c}{e}\sqrt{e^2 - 1}$, an equation for the hyperbola can be written

(1)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

The hyperbola has the same symmetries as the ellipse. The formulas for foci, vertices, and directrices are also the same; these are summarized for the various simple cases in Figure 7-6.

Unlike the ellipse, the hyperbola is not a bounded curve. In part (a) of Figure 7-6, for example, we see that if we take increasingly large values for x , the corresponding values for y are increasingly large in absolute value. On the other hand, there are values of x (in this case $-a < x < a$) for which there are no real values of y . If we solve (1) for y we get

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} .$$

For very large values of x , the values of y in the first quadrant are very nearly equal to $\frac{b}{a}x$ (corresponding comments apply in the other quadrants). Thus we see intuitively that for values of x that are sufficiently large in absolute value, the distance between a point on the curve and the line with equation $y = \frac{b}{a}x$ (or $y = -\frac{b}{a}x$) can be made arbitrarily small. Thus these lines are asymptotes of the hyperbola; in Figure 7-6 they are marked A and A' . You may wish to refer to Section 6-3 where there is a detailed discussion of the asymptotes of a particular hyperbola; it applies here.

To make a sketch of a hyperbola we first locate the vertices, and then draw the asymptotes. They are drawn easily since they are diagonals of the rectangle with sides $2a$ and $2b$, located as in Figure 7-6. The segment $\overline{VV'}$, of length $2a$, is called the transverse (or major) axis of the hyperbola. (The line segment joining the points $(0,b)$ and $(0,-b)$, of length $2b$, of part (a) of Figure 7-6 is sometimes called the conjugate axis.) From the relationship $c^2 = a^2 + b^2$, we see that the length of the diagonal of the rectangle is also the distance between the foci. We may use this fact to locate the foci.

Conjugate hyperbolas are concentric hyperbolas with the roles of the transverse and conjugate axes interchanged. The equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represent conjugate hyperbolas. As shown in Figure 7-7, they have the same asymptotes, and their foci lie on a circle with center at the center of the curves.

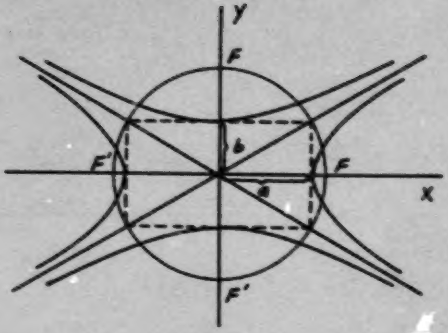


Figure 7-7

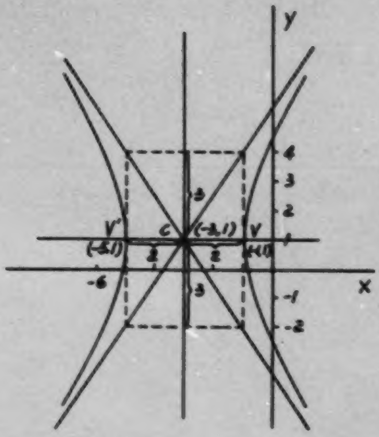
A hyperbola is called equilateral (or rectangular) if the transverse and conjugate axes are equal. In this case the rectangle we have used in sketching is a square, and the asymptotes (which are diagonals) are perpendicular. You may have studied the family of equilateral hyperbolas with equation $xy = k$. These are hyperbolas with the coordinate axes as asymptotes.

For any point of a hyperbola, the absolute value of the difference of its distances from two fixed points is a constant. This property is sometimes used to define a hyperbola; it has applications in range finding and LORAN (Long Range Navigation). Both of these use intersections of families of hyperbolas. As with the ellipse, a tangent at any point of a hyperbola makes equal angles with radii drawn to the foci; for the hyperbola, however, the radii are on opposite sides of the tangent.

Example. Find the equations of the asymptotes of the hyperbola with equation $9x^2 - 4y^2 + 54x + 8y + 41 = 0$. Sketch the curve and its asymptotes.

Solution. We rewrite the equation, following the same procedure as in Example 1 in Section 7-7, getting the equation

$$\frac{(x+3)^2}{2^2} - \frac{(y-1)^2}{3^2} = 1 .$$



This is in form (b) of Figure 7-6, with transverse axis having a length of 4 , the conjugate axis 6 ; the center is $C = (-3,1)$. To obtain the equations of the asymptotes, we write

$$\frac{(x+3)^2}{2^2} - \frac{(y-1)^2}{3^2} = 0$$

or $3x + 2y + 7 = 0$ and $3x - 2y + 11 = 0$. To make the sketch, we locate the center C , draw through C lines parallel to the coordinate axes, and mark off on them the lengths of the semi-axes. Next we draw the rectangle, its diagonals give the asymptotes, and we can sketch the curve.

Exercises 7-8

1. Write an equation of a hyperbola with semi-axes 2 and 3 , center at the origin, and transverse axis on the x-axis. Find the eccentricity, the coordinates of the vertices and foci, and equations of the directrices and asymptotes. Sketch the curve.
2. Repeat Exercise 1, but this time let the transverse axis be on the y-axis.
3. Write an equation of a hyperbola with center (-2,3) , semi-axes 4 and 3 , and transverse axis parallel to the x-axis. Find the eccentricity, coordinates of vertices and foci, and equations of directrices and asymptotes. Sketch the curve.
4. Repeat Exercise 3, but this time have the transverse axis parallel to the y-axis.

5. For each hyperbola whose equation is given, find the eccentricity and the length of the semi-axes; the coordinates of center, foci, and vertices; the equations of the directrices and asymptotes. Sketch the curves.

   (a) $x^2 - y^2 = 4$

   (b) $y^2 - x^2 = 4$

   (c) $4x^2 - 9y^2 = 36$

   (d) $144y^2 - 25x^2 = 3600$

   (e) $x^2 - 4y^2 - 4x + 24y - 16 = 0$

6. For each part of Exercise 5, write an equation of the conjugate hyperbola.

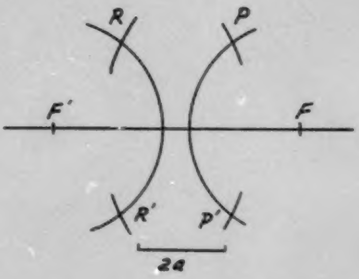
7. Find an equation of the locus of a point such that the absolute value of the difference of its distances from the points $(5,0)$ and $(-5,0)$ is 6.

8. Find an equation of the locus of a point such that the absolute value of the difference of its distances from the points $(1,1)$ and $(-1,-1)$ is 2. What is the eccentricity of this curve?

9. Prove that a hyperbola is the locus of a point such that the absolute value of the difference of its distances from two fixed points is a constant which is less than the distance between the fixed points.

10. What is an appropriate definition of the latus rectum of a hyperbola? Find a formula for the length of the latus rectum of a hyperbola; check that your formula applies in all four cases of Figure 7-6.

11. Construct some points of a hyperbola as follows. Select fixed points F, F' and a length $2a$ $\big(2a < d(F,F')\big)$. With F as center and any desired radius r, describe an arc. With F' as center and radius of length $r + 2a$, describe an arc intersecting the first arc at points P and P'. Then use F' as a center with radius r, and F with radius $r + 2a$, obtaining points R and R'. Thus for a particular choice of r, four points can be located. Why do the points so located lie on a hyperbola?



12. Prove that the equations $x = a \sec \theta$, $y = b \tan \theta$ are a parametric representation of a hyperbola.

13. See if you can devise a method of constructing a hyperbola which uses the equations in Exercise 12. (Hint: See Section 5-4.)

14. Find equations of the equilateral hyperbolas through the point $(3,-7)$,

    (a) with the coordinate axes as asymptotes.

    (b) with axes of the hyperbola along the coordinate axes.

15. Just as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ was considered an equation of a degenerate ellipse, we may speak of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ as the equation of a degenerate hyperbola. What is the locus in this case?

7-9. Summary

A conic section is the intersection of a plane and a right circular cone; it is a circle, ellipse, parabola, hyperbola, or, in a degenerate case, a point, line, or pair of lines.

In polar coordinates a circle with center at the origin has the equation $r = k$. Any other conic section may be defined as the locus of points in a plane such that for each point the ratio of its distance from a given point in the plane to its distance from a given line in the plane is a constant e, called the eccentricity. Such a conic, if the center is at the pole and directrix perpendicular to the polar axis and p units to the left of it, has the equation

$$r = \frac{ep}{1 - e \cos \theta}, \qquad \text{representing} \quad \left\{ \begin{array}{l} \text{a parabola if } e = 1, \\ \text{an ellipse if } 0 < e < 1, \\ \text{a hyperbola if } e > 1. \end{array} \right.$$

The equations that relate polar and rectangular coordinates were used to find corresponding rectangular equations. These were seen to be equivalent to the equations developed in earlier work in algebra. Since the information about the conics in rectangular form is summarized at the beginning of the sections (7-5 through 7-8) dealing with each type, it is not repeated again here.

Conic sections have wide usefulness in theoretical work in mathematics and science, and in applications to a great variety of problems in science and industry; it has been possible to mention only a few here.

With this chapter we conclude, for the time being, our study of the analytic geometry of two-space. We shall take up next the analytic geometry of three-space. Later, if time permits, there may be an opportunity to return again to conic sections in order to consider the general problem of showing that all equations of second degree in x and y have loci which are conic sections, and then to relate the corresponding algebraic and geometric properties.

Review Exercises

1. Sketch the graph of each of the following equations. Identify each conic section, and give the appropriate information (foci, vertices, center, eccentricity, directrices, asymptotes, etc.) .

(a) $3r - 2 = 0$

(b) $r = 2 \cos \theta$

(c) $r = \dfrac{8}{1 - \cos \theta}$

(d) $r = \dfrac{4}{2 - 3 \cos \theta}$

(e) $2 - \cos \theta = \dfrac{3}{r}$

(f) $r = \dfrac{12}{3 - 3 \cos \theta}$

(g) $4r = 3r \cos \theta + 24$

(h) $r = 4 - r \sin \theta$

(i) $r = 3 + 2r \cos \theta$

(j) $x^2 - 4x + y^2 + 6y + 13 = 0$

(k) $3x^2 - 2y^2 = 6$

(l) $y^2 + 8x - 6y + 25 = 0$

(m) $25x^2 + 36y^2 + 100x + 288y - 224 = 0$

(n) $3x^2 + 5y^2 - 6x + 20y + 8 = 0$

(o) $x^2 + y^2 - 6x + 10y + 34 = 0$

(p) $x^2 - 3y^2 + 8x - 6y - 14 = 0$

(q) $144x^2 - 25y^2 + 576x + 150y - 3249 = 0$

2. Write an equation for each of the following and sketch the graph.

   (a) A parabola with vertex $(0,0)$ and focus $(-5,0)$.

   (b) A parabola with vertex $(7,6)$ and directrix $y = -2$.

   (c) A circle with radius 5 and tangent to both axes.

   (d) A circle with center $C = (1,-4)$ and passing through $(3,-2)$.

   (e) A circle tangent to the line $x - 2y - 2 = 0$, passing through the point $(-2,0)$, and with center on the y-axis.

   (f) A circle passing through the points $(0,4)$, $(6,6)$ and $(-2,-10)$.

   (g) An ellipse with center $(2,3)$, a vertex $(5,3)$, and a directrix $x = -4$.

   (h) An ellipse with a focus $(-3,5)$, and directrices $y = 6$ and the x-axis.

   (i) A hyperbola with foci $(-1,1)$ and $(5,1)$, and a vertex $(0,1)$.

   (j) A hyperbola with asymptotes $3x - 4y = 0$, $3x + 4y = 0$, and passing through the point $(3,5)$.

   (k) A parabola with axis parallel to the y-axis, passing through the points $(2,11)$, $(0,5)$ and $(-1,8)$.

3. Find an equation of the locus of a point whose distance from the point $(-1,4)$ is 2 units more than its distance from the line $y + 2 = 0$.

4. Find an equation of the locus of the center of a circle which is tangent to the line $x = 3$ and passes through $(1,-1)$. Explain from geometric considerations why this locus must be a parabola.

5. Find the eccentricity of an ellipse whose major axis is twice the length of its minor axis.

6. Prove that the equations $x = a \cos \theta$, $y = b \sin \theta$ are a parametric representation of an ellipse.

7. Find an equation of the locus of a point which moves so that its distance from the point $(0,2)$ is one-half its distance from the point $(3,1)$.

8. Prove that the product of the distances from any point on a hyperbola to the asymptotes is a constant.

9. (a) If the ratio of the length of the conjugate axis to the length of the transverse axis of a hyperbola is 2 , what is the eccentricity?

   (b) If the ratio is k , find a formula for e .

10. (a) Show that $x = \dfrac{tr}{\pm\sqrt{1 + t^2}}$, $y = \dfrac{r}{\pm\sqrt{1 + t^2}}$ are parametric equations of a circle. (These equations are sometimes useful in calculus.)

(b) What is the graph of the equations in (a) if only the positive signs before the radicals are used? If only the negative signs?

(c) Show that these parametric equations do not represent the points $(r,0)$ and $(-r,0)$. Since this is the case, what would be a more precise way to state (a) in this exercise?

11. Prove that, for the conjugate hyperbolas whose equations are

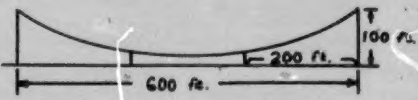
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

the sum of the squares of the reciprocals of the eccentricities is one.

12. A curve is defined by the parametric equations $x = a + k \cos \theta$, $y = b + k \sin \theta$, where a , b , and k are arbitrary constants $(k \neq 0)$. Find an equation of the curve in standard rectangular form and identify it. What is the significance of the requirement that k not be zero?

13. An archway is in the shape of a semi-ellipse. The distance across the base of the archway is 30 feet, and its maximum height from the base is 20 feet. What should be the limit on the height of vehicles using a centrally placed 20-foot wide road under the archway? (The posted limit is such that a vehicle of that height, at the edge of the road but not off the road, will have clearance.)

14. The cable of a suspension bridge hangs in the form of a parabola from supporting towers 600 feet apart. The points where the cable is suspended from the towers are 100 feet above the roadway, and the lowest part of the cable is 10 feet above the roadway. If there are supporting structures to the cable from the two points on the roadway each 200 feet from the base of the towers, how high must these supporting structures be?



15. Prove that the product of the focal radii from a point on an equilateral hyperbola is equal to the square of the distance from the point to the center.

16. (a) Write an equation of the family of ellipses with the origin as center, major axis along the x-axis, and eccentricity equal to $\frac{3}{5}$.

    (b) Write an equation for the member of this family with the length of the minor axis equal to 12 .

    (c) Write an equation for the member of this family which passes through the point $(4,\frac{12}{5})$.

17. Prove the following statements analytically.

    (a) A radius perpendicular to a chord bisects the chord.
    (b) The perpendicular from any point of a circle to a diameter is the mean proportional between the segments of the diameter.
    (c) The locus of a point such that its distance from one fixed point is a constant multiple of its distance from a second fixed point is a circle. (What restriction must there be on the value of the constant for this to be a correct statement?)

Challenge Problems

1. Prove that in a hyperbola an asymptote, a directrix, and a line from the corresponding focus perpendicular to the asymptote are concurrent.

2. On a map marked with a rectangular grid using a mile as a unit, three listening posts are at $A = (0,0)$, $B = (2,0)$, and $C = (0,4)$. An explosion is heard at A 5 seconds after it is heard at B , and 8 seconds after it is heard at C . Where did the explosion take place? (Use 0.2 mile per second as the speed of sound. Find equations of the two loci involved, and find the appropriate intersection either by graphing or by using the equations of the asymptotes. Do you think that it is sufficiently accurate to assume that the asymptotes meet at the point you want?)

3. A taxpayer changes his residence because of a change in his place of work. For his moving expenses to be allowed as a deduction under the Revenue Act of 1964, it is necessary (among other requirements) that his new principal place of work be "at least 20 miles farther from his former residence than was his former principal place of work."

   Suppose a man's new employment is at a place 30 miles from where he was previously employed. Let $P = (x,y)$ represent the location of his old home. Write in analytic form the condition under which the man would be entitled to deduct moving expenses to a new home. (Suggestion: If W_1 and W_2 are points representing the old and new places of employment respectively, let $\overleftrightarrow{W_1W_2}$ be the x-axis, and let the midpoint of $\overline{W_1W_2}$ be the origin.)

4. For the parabola $r = \frac{6}{1 - \cos\theta}$, prove the reflective property, that is, the tangent to the parabola at the point $P = (r,\theta)$ makes equal angles with the polar radius $\overline{OP}$ and the line through P parallel to the polar axis.

5. Prove analytically that, in any triangle, the midpoints of the sides, the feet of the altitudes, and the points halfway between the vertices and the orthocenter lie on a circle. This is called the nine-point circle.

Chapter 8

THE LINE AND THE PLANE IN 3-SPACE

8-1. The Extension to 3-Space.

To this point in our study we have sought analytic representations of subsets of a plane; in turn we have sketched the loci, or graphs, of both algebraic and vector relationships with the assumption, usually tacit, that their geometric interpretation was confined to a plane or a line.

Our previous experience in geometry has been largely in a plane; even when we did consider geometric configurations in space, we frequently pursued our investigations in only one or two planes.

It is easier to analyze loci in a plane, but we live in a world of three dimensions. If we are to apply our geometric knowledge to physical problems, we must be able to extend our concepts to 3-space.

In this chapter and the next we shall consider the basic extension to 3-space of the ideas which we have already developed; we shall even suggest how repetition of this process leads to mathematical structures with more dimensions, which are called spaces, even though we cannot possibly visualize them.

In this chapter we shall be extending some of the ideas of Chapters 2 and 3 to 3-space; you might want to review these chapters briefly before you continue. We assume that you have had some experience with rectangular coordinate systems in 3-space, but we shall reconstruct the development. We shall consider the analytic representations of lines and planes, and we shall make suggestions on sketching to help you visualize their graphs. The extension of vectors to spaces of higher dimension is surprisingly easy; this is another reason for the favor vectors find in contemporary analysis.

One thing you might keep in mind. The locus of a condition depends upon the space to which it is applied. We have already seen that the equation $x = 1$ describes both a point on a line and a line in a plane. Here we shall see that it also describes a plane in 3-space. In spaces of higher dimension it would be subject to still other interpretations. In general, analytic representations describe loci in any space which has at least as many dimensions as the analytic representation has independent variables. To describe the locus we must first know the number of dimensions of the space in which it occurs.

8-2. *A Coordinate System for 3-Space.*

In Sections 2-1 and 2-3 we discussed rectangular coordinate systems on a line and in a plane. Now we shall indicate how a similar coordinate system can be introduced into 3-space.

We begin by selecting an arbitrary point 0 in space and three mutually perpendicular lines through 0 . The point 0 is called the *origin* of the coordinate system and the lines are called the *x-*, *y-*, and *z-axes*. On each axis we set up a linear coordinate system with point 0 as its origin. The plane determined by the x- and y-axes is called the *xy-plane*. The *xz-* and *yz-planes* are defined similarly. The three are called the *coordinate planes*. Let P be any point in space. Let a be the coordinate of the projection of P on the x-axis. a is called the *x-coordinate* of P . The *y-* and *z-coordinates*, say b and c respectively, are defined similarly. To the point P we assign the *ordered triple* (a,b,c) of coordinates. Just as in the plane, the correspondence between points and ordered sets of coordinates is one-to-one. The coordinate planes divide space into eight regions called, not unnaturally, *octants*. Usually only one of them is numbered, and it is called the first and is the one in which all the coordinates of every point are positive.

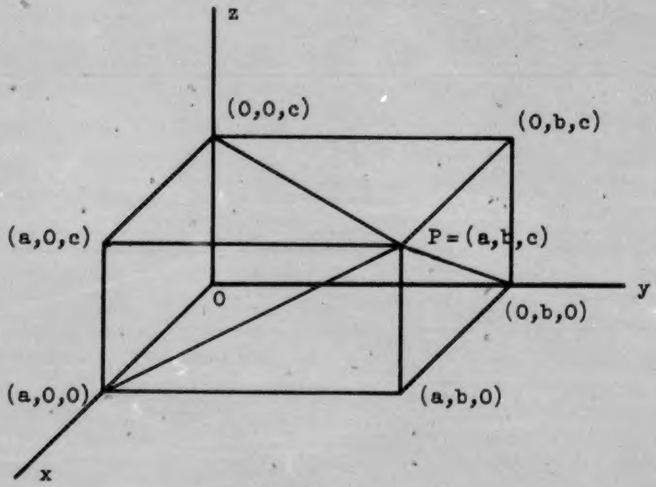


Figure 8-1

The point $(a,b,0)$ is called the projection of (a,b,c) on the xy-plane. The point $(a,0,0)$ is called the projection of (a,b,c) on the x-axis, and so forth.

The configuration of axes shown in Figure 8-1 is called a right-handed system because a 90° rotation of the positive side of the x-axis into the positive side of the y-axis will advance a right-handed screw along the positive side of the z-axis. We shall use this system in drawings in this text. If the locations of the x- and y-axes are interchanged, as you will find that they are in some texts, the system is left-handed.

Distance Between Two Points: We may use the Pythagorean Theorem to develop a formula for the distance between two points in space. If the points are $P_0 = (x_0,y_0,z_0)$ and $P_1 = (x_1,y_1,z_1)$, the distance between them is

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \,. \tag{1}$$

Points of Division: An extension to 3-space of the method used in Section 2-3 to obtain the coordinates of the point which divides a line segment in the ratio $\frac{c}{d}$ gives us, for the segment P_0P_1,

$$x = \frac{dx_0 + cx_1}{c + d}$$

$$y = \frac{dy_0 + cy_1}{c + d} \tag{2}$$

$$z = \frac{dz_0 + cz_1}{c + d}$$

In the special case when $c = d$, we have the midpoint, with

$$x = \frac{x_0 + x_1}{2}$$

$$y = \frac{y_0 + y_1}{2} \tag{3}$$

$$z = \frac{z_0 + z_1}{2}$$

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Exercises 8-2

1. Draw a sketch showing each of the following points in space:

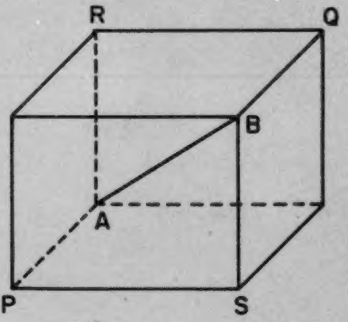
(a)	(1,2,1)	(e)	(-1,-1,2)
(b)	(-2,1,1)	(f)	(-1,-2,-1)
(c)	(2,0,-1)	(g)	(-3,1,-1)
(d)	(1,-1,2)	(h)	(1,-1,-2)

In Exercises 2, and 3, $P = (1,2,3)$, $Q = (-3,2,1)$, and $R = (2,-3,1)$.

2. Find $d(O,P)$, $d(O,Q)$, $d(P,R)$, and $d(Q,R)$.

3. Find the midpoints of $\overline{OP}$ and $\overline{PR}$.

4. (a) Draw $\overline{AB}$, about 3 inches long oblique to the edge of your paper. Consider $\overline{AB}$ as drawn from the rear lower left to the front upper right corner of a rectangular solid. Next draw oblique segments from A to P and from B to Q equal in length and parallel but with opposite sense of direction. If, as is usually the case, the solid is to be oriented with respect to rectangular coordinate axes, make $\overline{AP}$ and $\overline{BQ}$ parallel to the x-axis. Then draw a rectangle with horizontal and vertical sides and with P and B as opposite vertices; this is the front face. The back face is another rectangle with A and Q as opposite vertices. Two more segments complete the figure.



(b) Now start again with the same kind of diagonal segment $\overline{AB}$, but consider it drawn from the front lower left to the rear upper right, and draw the new solid. This time reverse the directions of $\overline{AP}$ and $\overline{BQ}$. Now A and Q are in the front face and B and P are in the back face.

5. The origin and the point $P = (3,5,4)$ are the opposite corners of a rectangular box that has three of its edges along the axes. Draw the box and give the coordinates of its other vertices.

6. Repeat Exercise 5, using $P = (-5,4,-3)$.

7. Given: $P_1 = (2,-3,4)$ and $P_2 = (-1,3,-2)$

   (a) Make a drawing which shows P_1, P_2, and $\overline{P_1P_2}$.

   (b) Write the coordinates of the points which are the projections of P_1 and P_2 on each of the axes and on each of the coordinate planes.

   (c) Find the length of $\overline{P_1P_2}$ and the length of its projections on the axes and on the coordinate planes.

8. Repeat Exercise 7, using $P_1 = (-3,5,7)$ and $P_2 = (3,0,-3)$.

9. If $P_1 = (3,-4,6)$ and $P_2 = (-2,3,-2)$ find the coordinates of point P on $\overleftrightarrow{P_1P_2}$ if

   (a) P is the midpoint of $\overline{P_1P_2}$.

   (b) $d(P_1,P) = \frac{1}{2}d(P,P_2)$

   (c) $d(P_1,P) = \frac{3}{5}d(P,P_2)$

   (d) $d(P_1,P) = \frac{5}{3}d(P,P_2)$

   (e) $d(P_1,P) = \frac{3}{5}d(P_1,P_2)$

   (f) $d(P_1,P) = \frac{5}{3}d(P_1,P_2)$

10. In triangle ABC, $A = (2,4,1)$, $B = (1,2,-2)$ and $C = (5,0,-2)$. Find the lengths of the sides of this triangle and decide what kind of triangle it is.

Challenge Problem

We introduced a coordinate system in 3-space by selecting three mutually perpendicular lines through an arbitrary point. Show that this is possible.

8-3. Parametric Representation of the Line in 3-Space.

Our discussion in Section 5-6 of the parametric representation of a line in a plane generalizes quite easily to 3-space. Let $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$ be two points in space and let L be the line through them.

Assume for the time being that L is not parallel to or lying in any coordinate plane. Then P_0 and P_1 cannot both lie in the xy-plane and we let P_1 be one which does not. Hence P_0, P_1, and $(x_1, y_1, 0)$ are not collinear and determine a plane M containing L. M intersects the xy-plane in a line L' called the projection of L on the xy-plane. Since the line containing P_1 and $(x_1, y_1, 0)$ is perpendicular to the xy-plane, plane M is perpendicular to the xy-plane. Hence the line from P_0 perpendicular to the xy-plane (and thus intersecting it in the point $(x_0, y_0, 0)$) lies in plane M and is a point of L, the line of intersection.

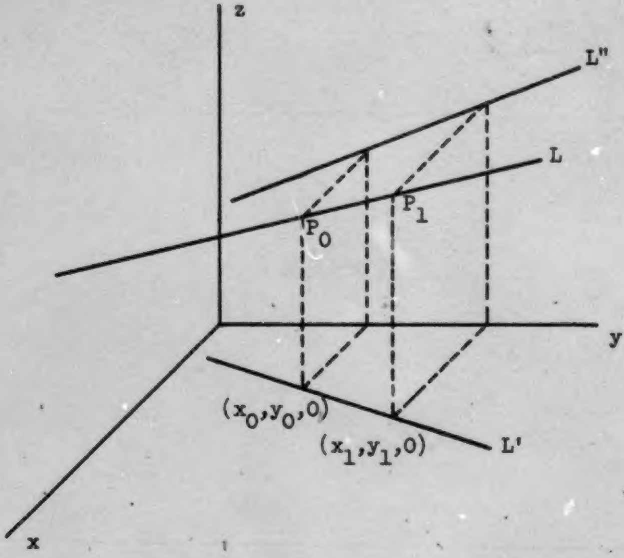


Figure 8-2

From our previous discussion, we know that L' has the parametric equations

$$(1)\qquad \begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) . \end{aligned}$$

We would have a parametric representation for L very similar to the one we obtained for a line in a plane if we could show that if $P = (x, y, z)$ is on L

$$z = z_0 + t(z_1 - z_0) .$$

Clearly

$$z = z_0 + s(z_1 - z_0)$$

for suitable s. The question is, is s equal to t? That it is can be proved as follows. Let L'' be the projection of L on the yz-plane. Then in this plane L'' has the parametric representation

$$\begin{aligned} y &= y_0 + s(y_1 - y_0) \\ z &= z_0 + s(z_1 - z_0) . \end{aligned} \tag{2}$$

From (1) and (2) it follows that for each point $P = (x,y,z)$ of L, $s = t$, and hence L has the parametric representation

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \\ z &= z_0 + t(z_1 - z_0) . \end{aligned} \tag{3}$$

We leave it to the student as an exercise to prove that (3) represents L even if L is in or parallel to a coordinate plane.

To save writing, let $\ell = x_1 - x_0$, $m = y_1 - y_0$, and $n = z_1 - z_0$. We call (ℓ,m,n) an ordered triple of direction numbers for L. If $c \neq 0$ the equations

$$\begin{aligned} x &= x_0 + c\ell t \\ y &= y_0 + cmt \\ z &= z_0 + cnt \end{aligned}$$

also represent L. Thus it is natural to extend the definition of equivalence of ordered pairs of direction numbers for a line in a plane to ordered triples of direction numbers for a line in space. Two such ordered triples are said to be equivalent if corresponding numbers are proportional.

Let L and L' be the lines with parametric equations

$$L: \begin{cases} x = x_0 + \ell t \\ y = y_0 + mt \\ z = z_0 + nt \end{cases} \qquad L': \begin{cases} x = \ell t \\ y = mt \\ z = nt \end{cases}$$

and assume L does not go through the origin. Then, as we shall prove, L and L' are parallel.

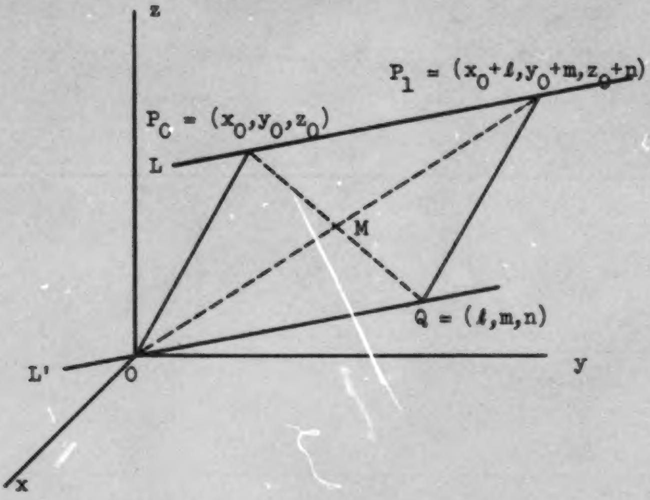


Figure 8-3

Let $P_1 = (x_0 + \ell, y_0 + m, z_0 + n)$, $Q = (\ell, m, n)$. Then P_1 and $P_0(x_0, y_0, z_0)$ are on L ; O and Q are on L' . The midpoint of $\overline{OP_1}$ is

$$M = \left(\frac{x_0 + \ell}{2}, \frac{y_0 + m}{2}, \frac{z_0 + n}{2}\right),$$

M is also the midpoint of $\overline{P_0Q}$. Thus OP_0P_1Q is a plane quadrilateral whose diagonals bisect each other and hence is a parallelogram. It follows that L and L' are parallel. The following theorem is an almost immediate consequence of our argument.

THEOREM 8-1. Two distinct lines L and L' are parallel if and only if any triple of direction numbers for L is equivalent to any one for L' .

As in the plane, a set of direction numbers for a line can be used to establish a direction on the line. Let (ℓ, m, n) be a triple of direction numbers for the line L . If $P_0 = (x_0, y_0, z_0)$ is a point on L , L has the representation

$$x = x_0 + \ell t$$
$$y = y_0 + mt$$
$$z = z_0 + nt .$$

The positive ray (on L) with endpoint P_0 is the set of points consisting of P_0 and all points of L given by positive values of t . If P_1 is another point of L , the positive ray with endpoint P_1 points in the same direction as the one with endpoint P_0 in the sense that their intersection is one of them. If $c > 0$, the triple $(c\ell, cm, cn)$ of direction numbers for L establishes the same positive direction on L as does the triple (ℓ,m,n) .

If (ℓ,m,n) is a triple of direction numbers for L , the triple

$$(\lambda,\mu,\nu) = \left(\frac{\ell}{\sqrt{\ell^2+m^2+n^2}}, \frac{m}{\sqrt{\ell^2+m^2+n^2}}, \frac{n}{\sqrt{\ell^2+m^2+n^2}}\right)$$

is of particular importance. Such a triple is sometimes called a normalized triple. Note that $\lambda^2+\mu^2+\nu^2 = 1$. Let us assume first that L goes through the origin. The point $P = (\lambda,\mu,\nu)$ lies on L and $d(O,P) = 1$. Figures 8-4a and 8-4b show the situation when $\lambda > 0$, $\mu > 0$, $\nu > 0$ and the situation when $\lambda < 0$, $\mu > 0$, $\nu > 0$ respectively. In both cases, $\mu = \cos\beta$, where β is the angle determined by the positive ray on L with endpoint O and the positive half of the y-axis. α and γ are defined similarly, with the positive halves of the x- and z-axes, respectively, replacing the positive half of the y-axis.

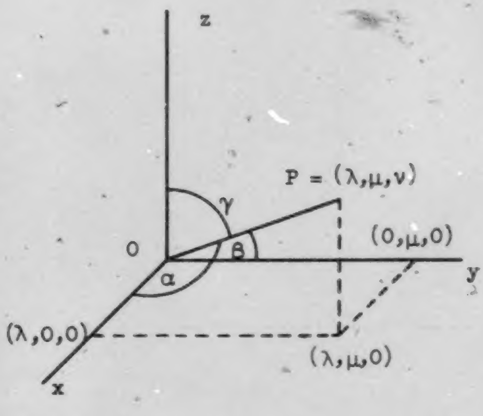


Figure 8-4a

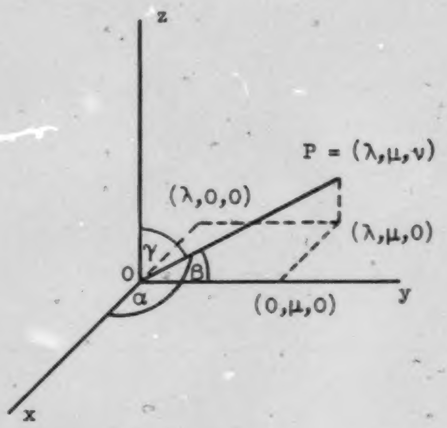


Figure 8-4b

If L is the x-axis, then any triple of direction numbers for it has the form $(\ell,0,0)$. If $\ell > 0$, the positive ray with endpoint 0 is the positive half of the x-axis and $\cos \alpha = 1$. If $\ell < 0$, the positive ray on L with endpoint 0 is the negative half of the x-axis and $\cos \alpha = -1$. Similarly, if L is the y-axis, $\cos \beta = \pm 1$ depending on the algebraic sign of m , and if L is the z-axis, $\cos \gamma = \pm 1$ depending on the algebraic sign of n . The student should consider the other possible combinations of signs for λ , μ , and ν , to make sure that in every case $\lambda = \cos \alpha$, $\mu = \cos \beta$, and $\nu = \cos \gamma$. The angles α , β , and γ are called direction angles of the line L with its direction determined by the ordered triple (ℓ,m,n) of direction numbers. Their cosines are called the direction cosines. If we determine the direction of L by means of the triple $(c\ell,cm,cn)$ of direction numbers, with $c < 0$, and if α' , β' , and γ' are the new direction angles, then α and α' are supplementary angles, as are β and β' , and γ and γ' .

Finally, let L be a line which does not pass through the origin, and let (ℓ,m,n) be an ordered triple of direction numbers for L . Let L' be the line through the origin parallel to L , and let the direction on L' be determined by the triple (ℓ,m,n) of direction numbers. Then we define the direction angles and cosines of L to be the corresponding ones for L' .

Notice that throughout this discussion we do not define direction angles or direction cosines for a line, but only for a line which has been assigned a direction by means of a triple of direction numbers.

In Section 2-3 we derived a parametric representation of points on a line from their symmetric representation. Something similar can be done with a parametric representation of a line in space. Let L be the line with parametric equations

$$\begin{aligned} x &= x_0 + \ell t \\ y &= y_0 + mt \\ z &= z_0 + nt . \end{aligned} \tag{4}$$

Suppose that $\ell mn \neq 0$. Then we can eliminate t from any two of these equations by solving each one for t and setting the results equal to each other. Using the first two, we get

$$t = \frac{x - x_0}{\ell} = \frac{y - y_0}{m} .$$

Using the first and third, we get

$$t = \frac{x - x_0}{\ell} = \frac{z - z_0}{n} .$$

Combining the last two results we get

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{n} . \tag{5}$$

These are called symmetric equations for L .

There remains the question of what we have achieved by eliminating t . Let t_0 be any real number and let

$$a = x_0 + \ell t_0$$

$$b = y_0 + m t_0$$

$$c = z_0 + n t_0 .$$

Then

$$\frac{a - x_0}{\ell} = \frac{b - y_0}{m} = \frac{c - z_0}{n} .$$

Thus if the point (a,b,c) is on the graph of (4) it is also on the graph of (5). If we let

$$t_0 = \frac{a - x_0}{\ell} = \frac{b - y_0}{m} = \frac{c - z_0}{n} ,$$

we find at once that the point (a,b,c) also lies on the graph of (4). Thus the graphs of (4) and (5) are identical.

Equations (5) are equivalent to any pair of the three equations

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$$

$$\frac{x - x_0}{\ell} = \frac{z - z_0}{n}$$

$$\frac{y - y_0}{m} = \frac{z - z_0}{n} .$$

Each of these is an equation of a plane. We shall discuss the significance of this particular set of three planes containing a line in the next section.

If at least one of the direction numbers for L vanishes we cannot write such symmetric equations for L. We can, however, eliminate t and obtain equations of two planes containing L. We leave this to the exercises.

You may have read of spaces of four or more dimensions. We are now in a position to give you some idea of what was meant. You have learned how to set up a one-to-one correspondence between the points in a plane and the ordered pairs of real numbers, and between the points in 3-space and the ordered triples of real numbers. Given a coordinate system, it is natural to speak of "the point $(2,3)$" or "the point $(3,2,-1)$." This suggests that we should define a *point in 4-space*, for example, to be an ordered quadruple of real numbers. Similarly, we define a *line in 4-space* to be the set of points in 4-space given by a set of parametric equations of the form

$$\begin{aligned} x &= x_0 + \ell t \\ y &= y_0 + mt \\ z &= z_0 + nt \\ w &= w_0 + pt . \end{aligned}$$

It can then be proved that there is one and only one "line" through two distinct "points." We can define the *distance* between $P_0(x_0,y_0,z_0,w_0)$ and $P_1(x_1,y_1,z_1,w_1)$ to be

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 + (w_1 - w_0)^2} .$$

We can define the *coordinate axes* to be the four "lines" through $(0,0,0,0)$ each of which passes through one of the "points" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. Many other geometric concepts you have studied can be generalized in this way, but that is beyond the scope of this course.

Example. If $A = (3,-1,4)$, $B = (-2,2,1)$ and $C = (2,3,-2)$,

(a) write parametric and symmetric representations for $\overleftrightarrow{AB}$, and

(b) write equations for the line through C parallel to $\overleftrightarrow{AB}$.

Solution.

(a) For parametric form (Equations (4)), we need a point on the line and direction numbers. We choose $A = (3,-1,4)$, and obtain direction numbers $(5,-3,3)$. Hence the line $\overleftrightarrow{AB}$ has as a parametric representation

$$\begin{aligned} x &= 3 + 5t \\ y &= -1 - 3t \\ z &= 4 + 3t . \end{aligned}$$

From the first two of these we get

$$t = \frac{x-3}{5} = \frac{y+1}{-3} .$$

From the last two we get

$$t = \frac{y+1}{-3} = \frac{z-4}{3} .$$

Combining the last two results, we have as symmetric equations for $\overleftrightarrow{AB}$

$$\frac{x-3}{5} = \frac{y+1}{-3} = \frac{z-4}{3}$$

(b) Since we have direction numbers for $\overleftrightarrow{AB}$, we can write immediately a parametric representation of a parallel line through C,

$$\begin{aligned} x &= 2 + 5t \\ y &= 3 - 3t \\ z &= -2 + 3t \end{aligned}$$

Exercises 8-3

In Exercises 1 to 3, $P = (1,2,3)$, $Q = (-3,-2,1)$, and $R = (2,-3,1)$.

1. Write parametric equations for the lines determined by the following conditions:

   (a) Through P , parallel to the x-axis
   (b) Through Q , parallel to the z-axis
   (c) Through P and Q
   (d) Through Q and R
   (e) Through O parallel to $\overleftrightarrow{PQ}$
   (f) Through O parallel to $\overleftrightarrow{QR}$
   (g) Through O and P
   (h) Through P , parallel to the xy-plane, and intersecting the z-axis
   (i) Through P parallel to $\overleftrightarrow{QR}$
   (j) Through R parallel to $\overleftrightarrow{PQ}$

2. Write an equation in symmetric form for each of the lines referred to in Exercise 1 (if it is possible to do so).

3. Write a set of normalized direction numbers for each of the lines described in Exercise 1.

4. Find two parametric representations of the line through each of the following pairs of points which establish opposite directions on the line. Find the coordinates of another point on each line.

   (a) $(1,1,-2)$ and $(0,-1,-1)$
   (b) $(-1,-1,-1)$ and $(-2,-1,1)$
   (c) $(4,2,1)$ and $(1,-2,4)$
   (d) $(-3,1,1)$ and $(1,2,-1)$

5. Find the two triples of direction cosines for each line in Exercise 1. Using a table of the values of the trigonometric functions, find the approximate value of each of the direction angles.

6. What are direction cosines for the axes?

7. Find direction cosines of a line that makes equal angles with the axes.

8. In each of the following parts determine whether the third point is on the line containing the first two.

   (a) $(1,1,-2)$, $(0,-1,-1)$, $(2,3,-2)$
   (b) $(1,0,1)$, $(-1,-1,-2)$, $(-7,-4,-11)$

9. Determine which, if any, of the lines determined by the following pairs of points are parallel.

(a) (1,1,-2) and (-1,2,3)
(b) (3,-1,2) and (-1,1,11)
(c) (1,-1,3) and (5,1,11)
(d) (-3,5,12) and (1,3,3)
(e) (2,-3,4) and (-2,-5,-6)
(f) (-1,0,1) and (1,-1,-4)

10. Write symmetric equations for the lines

$$L_1: \begin{cases} x = 2 + 3t \\ y = 1 - 2t \\ z = -1 - t \end{cases} \qquad L_2: \begin{cases} x = -1 + t \\ y = 2 + 2t \\ z = 4 - t \end{cases}$$

$$L_3: \begin{cases} x = 3 + 2t \\ y = -5 - 3t \\ z = 4t \end{cases} \qquad L_4: \begin{cases} x = 2 - t \\ y = -1 + 3t \\ z = -2 \end{cases}$$

11. Prove that if L has the parametric representation $x = x_0 + \ell t$, $y = y_0 + mt$, $z = z_0 + nt$, and if P_1 and P_2 are the points on L given by the values $t = t_1$ and $t = t_2$, then

$$d(P_1P_2) = \sqrt{\ell^2 + m^2 + n^2}\,|t_2 - t_1| .$$

Interpret this result in words, including the special case when the direction numbers are normalized.

12. Prove that Equations (3) represent L even if L is in or parallel to a coordinate plane.

Challenge Problems

1. Find equations of two planes which intersect in the line

$$\begin{cases} x = 2 \\ y = -1 + t \\ z = 2 + 3t . \end{cases}$$

Explain carefully how you know both the planes contain the line.

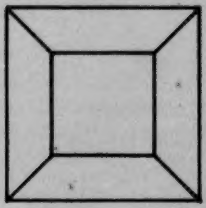
2. Find equations of two planes which intersect in the line

$$\begin{cases} x = 2 \\ y = -1 + t \\ z = -1 . \end{cases}$$

3. Find parametric equations for the "line" L through the "points" $P_0 = (x_0,y_0,z_0,w_0)$ and $P_1 = (x_1,y_1,z_1,w_1)$. Prove that if $P_2 = (x_2,y_2,z_2,w_2)$ is any other "point" on L, then the "line" through P_0 and P_2 contains P_1. Thus there is only one "line" through two given "points".

4. Let $P_0 = (x_0,y_0,z_0,w_0)$. Find the coordinates of the projections of P_0 on the coordinate axes, on the coordinate planes, and on the coordinate hyperplanes. (Before you can do the last part you will have to decide what it means.)

5. A cube in 3-space has an analog in 4-space which is called a tesseract. Make a three-dimensional "picture" of a tesseract. (It may help you to think about the sketch below, in which a cube is drawn in a plane.



The six faces of the cube, which are squares, are represented by two squares and four trapezoids.) In 3-space there is a relationship connecting the numbers of vertices, edges, and faces of a polyhedron. Try to discover this relationship by considering some simple cases. Try to find a corresponding theorem in 4-space.

8-4. The Plane in 3-Space.

In a plane, the set of points equidistant from two distinct points is a line; the equation of a line in 2-space is of first degree. In 3-space, the set of points equidistant from two distinct points is a plane. We review briefly the derivation of the equation of a plane; you may recall it from Intermediate Mathematics.

The point $P = (x,y,z)$ is equidistant from two distinct points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$, if

$$d(P_1,P) = d(P_2,P) ,$$

or

$$\sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} = \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} .$$

We square both members of the last equation and collect terms, obtaining

$$(1) \quad 2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z - \left((x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)\right) = 0 .$$

Since $d(P_1,P)$ and $d(P_2,P)$ are positive numbers, this argument can be reversed, and any point $P = (x,y,z)$ whose coordinates satisfy Equation (1) is equidistant from P_1 and P_2.

Equation (1) is a first-degree equation since the coefficients of x, y, and z are not all zero (they could all be zero only if P_1 and P_2 were the same point, but they are distinct).

Thus we have shown that the equation of a plane in three-space is a linear equation of the form

$$(2) \qquad ax + by + cz + d = 0 ,$$

where

$$a = 2(x_2 - x_1) , \; b = 2(y_2 - y_1) , \; c = 2(z_2 - z_1) ,$$

and

$$d = -\left((x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)\right) .$$

The proof of the converse--that every equation of the form (2) represents a plane--is left as an exercise.

We note that the coefficients of x, y, and z in Equation (1) are direction numbers of $\overleftrightarrow{P_1P_2}$, a line perpendicular to the plane; hence they are direction numbers of any normal to the plane. We shall extend this idea in Section 8-6. We also note that since $P_1 \neq P_2$, the coefficients a, b, and c are not all zero. The restriction on a, b, c is necessary. Let $a = b = c = 0$. If d is not zero, no triple (x,y,z) satisfies the equation, while if d is zero, every triple satisfies the equation. Neither one of these sets is a plane.

Let us consider certain first-degree equations in which some coefficients are zero. If the equation is of the form $ax = 0$ (or $x = 0$), it represents a plane in which the x-coordinate of every point is zero; clearly this is the yz-plane. In the same way, equations of the other coordinate planes are of the form $by = 0$ (or $y = 0$) and $cz = 0$ (or $z = 0$).

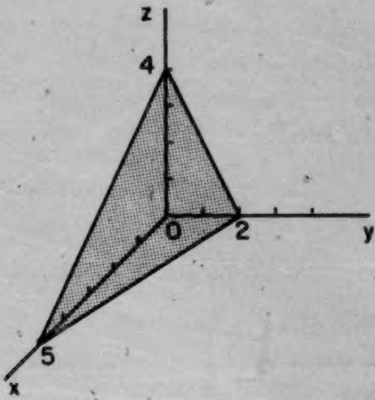
In general, we may find it helpful in visualizing a plane whose equation is given, and in drawing its graph, to find the traces. These are the intersections of the plane with the coordinate planes.

Example 1. Sketch the graph of $4x + 10y + 5z - 20 = 0$.

Solution. To find the trace in the xy-plane we let $z = 0$ in the equation of the plane, obtaining

$$4x + 10y - 20 = 0 \ .$$

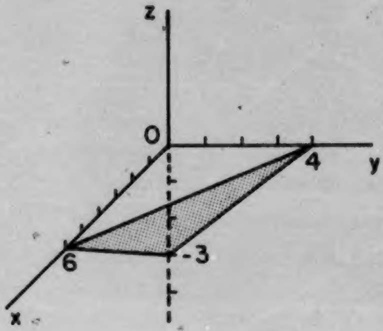
This is the equation of a straight line in the xy-plane.



In similar fashion, we find equations of the traces in the yz- and xz-planes ($10y + 5z - 20 = 0$ and $4x + 5y - 20 = 0$ respectively.) The graphs of these lines in the coordinate planes (or the parts of the graphs in one octant) suggest the graph of $4x + 10y + 5z - 20 = 0$.

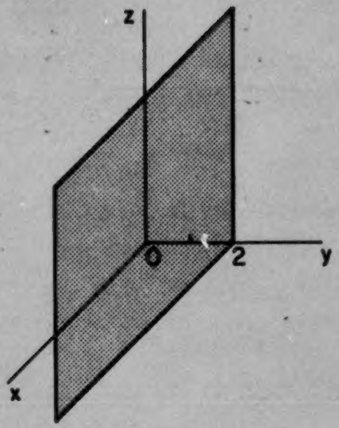
Example 2. Sketch the graph of $2x + 3y - 4z - 12 = 0$.

Solution. As in Example 2, we find equations of the traces in the xy-, yz-, and xz-planes ($2x + 3y - 12 = 0$, $3y - 4z - 12 = 0$, and $2x - 4z - 12 = 0$ respectively) and then make the sketch.



Example 3. Sketch the graph of $y - 2 = 0$.

Solution. We procede as before, drawing the graphs of $y = 2$, the equation of the traces in the xy- and yz-planes. There is no trace in the xz-plane; to make our representation compatible with our idea of a plane, we complete a parallelogram parallel to the xz-plane.



Since, if two different planes intersect, their intersection is a line, we can represent a line by the equations of any two different planes containing that line. With this in mind, let us look again at what we found in Section 8-3 as the symmetric equations for a line L ,

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{n} .$$

These equations are equivalent to any pair of the three equations

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$$

$$\frac{x - x_0}{\ell} = \frac{z - z_0}{n}$$

$$\frac{y - y_0}{m} = \frac{z - z_0}{n} ,$$

We know from the argument in Section 8-3 that each of the three planes contains L . Furthermore, each one lacks one of the variables. This means that each of the planes is perpendicular to one of the coordinate planes. This follows because, in the first of these three planes, for example, if (x_1,y_1,z_1) is a point in the plane, so also is (x_1,y_1,k) where k has any real value. Thus for any point of the plane, a line perpendicular to the xy-plane through that point is contained in the plane. These symmetric equations represent three planes, each containing the line and each perpendicular to a coordinate plane. These planes are called the projecting planes of L . They are special cases of the projecting cylinders of a curve which will be considered in Chapter 9.

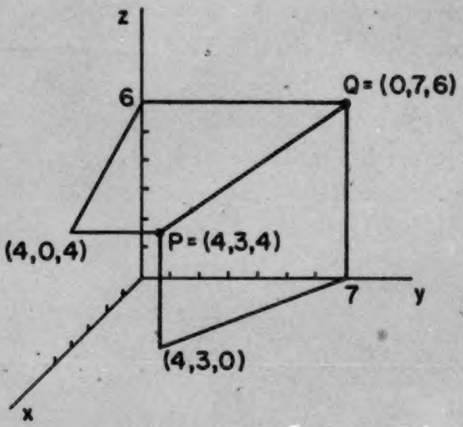


Figure 8-5

Example 4. Sketch the line with equations

$$\frac{x - 4}{2} = \frac{y - 3}{-2} = \frac{z - 4}{-1}$$

by using projecting planes.

Solution. We write the equations of two of the projecting planes,

$$\frac{x - 4}{2} = \frac{y - 3}{-2}$$

and

$$\frac{x - 4}{2} = \frac{z - 4}{-1} .$$

These equations may be rewritten as $x + y = 7$ and $x + 2z = 12$. We draw parts of the lines with these equations in the xy- and xz-planes, and complete the sketch as shown in Figure 8-5.

Now we turn to the problem of finding the distance between a point $P_0 = (x_0, y_0, z_0)$ and a plane M with equation

$$ax + by + cz + d = 0 .$$

There is a unique line N , containing P_0 , and normal to plane M . If N and M intersect at P_1 , the distance between P_0 and M , which we seek, is $d(P_0, P_1)$. We write parametric equations for N , using direction cosines; they are

$$x = x_0 + \lambda t$$
$$y = y_0 + \mu t$$
$$z = z_0 + \nu t .$$

Let t_1 represent the particular value t which gives the distance between P_0 and P_1 , the point in which N intersects M . Since P_1 is in M , its coordinates satisfy the equation for M ;

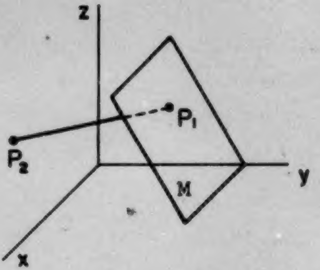


Figure 8-6

hence

$$a(x_0 + \lambda t_1) + b(y_0 + \mu t_1) + c(z_0 + \nu t_1) + d = 0 \ ,$$

or

$$(a\lambda + b\mu + c\nu)t_1 = -(ax_0 + by_0 + cx_0 + d) \ .$$

If we divide both members of this equation by $\sqrt{a^2 + b^2 + c^2}$ we get

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\lambda + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\mu + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\nu\right)t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} \ .$$

Since a , b , c are direction numbers for N , $\dfrac{a}{\sqrt{a^2 + b^2 + c^2}} = \lambda$, $\dfrac{b}{\sqrt{a^2 + b^2 + c^2}} = \mu$, and $\dfrac{c}{\sqrt{a^2 + b^2 + c^2}} = \nu$. We substitute λ , μ , ν , and obtain

$$(\lambda^2 + \mu^2 + \nu^2)t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} \ .$$

But, since λ , μ , and ν are direction cosines, $\lambda^2 + \mu^2 + \nu^2 = 1$; so

$$t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} \ ,$$

and (3)
$$d(P_0,P_1) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \ .$$

Example 5. Find the distances between $P = (1,-2,3)$ and planes $M_1 = \{(x,y,z) : 3x - 2y + z - 5 = 0\}$ and $M_2 = \{(x,y,z) : x + y = 0\}$.

Solution. Using Equation (3) , we find that

$$d(P_2,P_1) = \frac{|3(1) - 2(-2) + 1(3) - 5|}{\sqrt{9 + 4 + 1}} = \frac{5}{\sqrt{14}} \ ,$$

and

$$d(P_1,P_2) = \frac{|1(1) + 1(-2)|}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} \ .$$

Exercises 8-4

1. Write and simplify the equation of the locus of points equidistant from $A = (-2,3,5)$ and $B = (2,1,-3)$. Check your work by using a different method to find the equation of the plane which is the locus.

2. Follow the instructions in the first exercise, but use $A = (3,1,-4)$ and $B = (2,-3,1)$.

3. Find the intercepts and traces of the planes whose equations are given, and sketch the planes.

   (a) $6x + 4y + 3z - 12 = 0$
   (b) $2x + 5y + z - 10 = 0$
   (c) $4x - 2y - 5z - 10 = 0$
   (d) $3x - 2y + z + 6 = 0$
   (e) $3x - 4y - 12 = 0$
   (f) $5y - 8z + 20 = 0$
   (g) $3x - 6y + 2z = 0$
   (h) $3y - 5z = 0$
   (i) $x - 7 = 0$
   (j) $2z + 9 = 0$

4. Write an equation of the family of planes:

   (a) containing the origin
   (b) parallel to the xy-plane
   (c) parallel to the yz-plane
   (d) parallel to the z-axis
   (e) parallel to the x-axis
   (f) perpendicular to the xz-plane

5. Draw the line determined by the points $A = (5,1,3)$ and $B = (1,4,5)$ by

   (a) using the method described in Exercises 8-2, no. 4; and
   (b) drawing two of the projecting planes.

6. Repeat Exercise 5, using $A = (2,2,3)$ and $B = (0,5,5)$.

7. What is a set of direction numbers for a line perpendicular to the plane $M = \{(x,y,z) : 3x - 2y + 5z - 7 = 0\}$? Write the direction cosines for such a line.

8. Repeat Exercise 7 for the plane $M = \{(x,y,z) : 4x - y + 2 = 0\}$.

9. Find the distance from the point $P = (-1,2,2)$ to each of the planes with equations given in Exercise 3.

10. Repeat Exercise 9 but use the point $P = (1,4,-1)$.

11. Find an equation of the plane through the points

    (a) $(1,2,3)$, $(-1,-1,4)$, $(2,0,1)$
    (b) $(2,1,1)$, $(5,2,3)$, $(-1,-1,-1)$

12. Find an equation of a plane through P and parallel to M if

    (a) $P = (1,2,-3)$; $M = \{(x,y,z) : 3x - 2y + z - 7 = 0\}$

    (b) $P = (-1,2,2)$; $M = \{(x,y,z) : x - 2z + 3 = 0\}$

13. Show that if the x- , y- , and z-intercepts of a plane are a , b , and c respectively, an equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 .$$

14. Write an equation of the plane with x- , y- , and z-intercepts respectively

    (a) 1 , 3 , 4 ;

    (b) -2 , 5 , -3 .

15. Write an equation of a plane containing the point P and the intersection of planes M and N when

    (a) $P = (1,0,2)$, $M = \{(x,y,z) : x - 2y + z - 1 = 0\}$,
    $N = \{(x,y,z) : 2x + y + z + 1 = 0\}$.

    (b) $P = (3,1,-1)$, $M = \{(x,y,z) : x + 3y - 4 = 0\}$,
    $N = \{(x,y,z) : y - 2z + 3 = 0\}$.

16. Show that the four points $A = (1,2,1)$, $B = (2,-1,-4)$, $C = (0,1,2)$, $D = (2,3,0)$ are coplanar.

17. Find an equation of the plane containing the points:

    (a) (1,-1,1) , (2,0,0) , (-1,-1,2)

    (b) (1,3,5) , (2,1,2) , (0,-1,-1)

18. Prove that any equation of the form $ax + by + cz + d = 0$ represents a plane. (This is the converse of the proof at the beginning of this section.)

8-5. Vectors in Space; Components in 3-Space.

For vectors the extension to 3-space is not only natural, but also particularly easy. In your study of Chapter 3 you may have realized that the distinction between parallel and collinear vectors is not as clear as the distinction between parallel and collinear directed segments. Actually, there is no distinction. Because a vector is a set of equivalent directed segments, two vectors which have representatives on parallel lines also have representatives on the same line. In fact, a vector on a line has representatives anywhere on any line parallel to the given line. If $\hat{a}$ is a vector, every point in space is the initial point (or, for that matter, any other point on the line) of a representative of a . This is the basis for the Origin Principle and the Origin-Vector Principle.

For the same reason no two vectors may be noncoplanar. If the representatives of two vectors lie on skew (noncoplanar) lines, they not only have other representatives in a single plane, but also representatives in any other parallel plane. Furthermore, in such a plane they may be represented, of course, by origin-vectors.

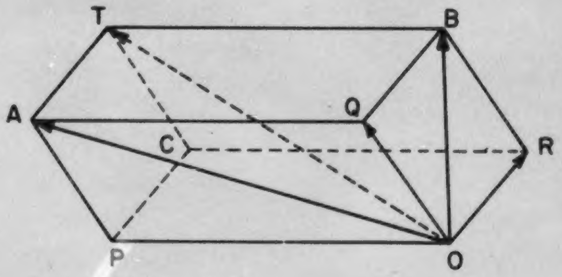
The definitions and properties of operations which involve no more than two vectors, such as addition, scalar multiplication, the distributive laws, and the inner product, apply in space, and may be interpreted geometrically in space. Theorems describing relations between two vectors also apply and may be interpreted in space. If at this point you will reread the definitions, principles, and theorems developed in Section 3-2 through Section 3-5 (pages 91-112), you will see that every statement and proof applies to vectors in space. The figures illustrate the situation in a plane, and in accordance with the Origin-Vector Principle our proofs are in terms of origin-vectors which are coplanar. As our discussion here indicates, our definition of vectors is such that a geometric relationship in space may often be described by vectors in a plane. In general, the vector description of a problem in space frequently may be reduced to a vector illustration in a plane. The illustration in the plane may serve as a simpler guide to the algebraic relations between the vectors. The results obtained may then be applied to the original problem in space. Of course, we must bear in mind that not all sets of vectors are coplanar.

As you reviewed the material in Chapter 3, you may have wondered whether the discussion above justifies the statement that Theorem 3-2, the associative property for vector addition, does apply in space. After all, the theorem states that $\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R}) = (\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R}$, and the three origin-vectors need not be coplanar. Strictly speaking, the assertion is valid, for vector addition is a binary operation; that is, we never add more than two vectors at a time. Therefore, as we perform each step of the proof, we are only adding vectors in a single plane, though the plane we work in may change from step to step in the proof as a whole. Still, the theorem is interesting and illustrative enough to consider as an example.

Example 1. Prove the associative property for vector addition:

$$\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R}) = (\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R} .$$

Proof. In the figure below we illustrate three noncoplanar origin vectors, $\overrightarrow{P}$, $\overrightarrow{Q}$, and $\overrightarrow{R}$. The segment $\overline{AQ}$ is drawn parallel and congruent to $\overline{PO}$ and the segment $\overline{RB}$ is drawn parallel and congruent to $\overline{OQ}$. Each of the quadrilaterals POQA and ORBQ are parallelograms, since in each two opposite sides are parallel and congruent. $\overline{BT}$ is drawn parallel and congruent to $\overline{AQ}$, and thus also to $\overline{PO}$.



$\overline{AT}$ is drawn. Since $\overline{TB}$ and $\overline{AQ}$ are parallel and congruent, quadrilateral AQBT is a parallelogram. Therefore, $\overline{AT}$ is parallel to $\overline{QB}$ and also to $\overline{OR}$. (If $\overline{CR}$ is drawn parallel and congruent to $\overline{PO}$, and $\overline{PC}$ and $\overline{CT}$ are also drawn, the entire figure is a parallelopiped, a prism whose base is a parallelogram region. However, we have not quite proved this here.) Since $\overline{PO}$ and $\overline{TB}$ are parallel and congruent, quadrilateral POBT is a parallelogram. Since $\overline{AT}$ and $\overline{OR}$ are parallel and congruent, quadrilateral ORTA is also a parallelogram.

We have now identified enough parallelograms to enable us to perform the vector additions required in the statement of the associative property.

The left member

$$\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R}) = \overrightarrow{P} + \overrightarrow{B} = \overrightarrow{T} ,$$

since ORBQ and POBT are parallelograms, and the right member

$$(\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R} = \overrightarrow{A} + \overrightarrow{R} = \overrightarrow{T} ,$$

since POQA and ORTA are parallelograms, thus

$$\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R}) = (\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R} .$$

Once a rectangular coordinate system has been introduced in 3-space, we have a one-to-one correspondence between the ordered triples of real numbers and the terminal points of origin-vectors. Thus, if the terminal point of the origin-vector $\overset{\frown}{A}$ has coordinates (a_1,a_2,a_3) , we may denote $\overset{\frown}{A}$ in component form by $[a_1,a_2,a_3]$, where a_1 , a_2 , and a_3 are the x- , y- , and z- components respectively.

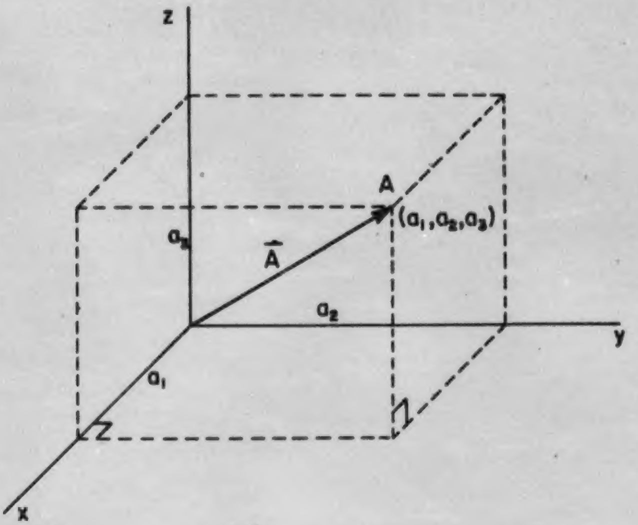


Figure 8-7

It follows from the definition that two vectors $\overset{\frown}{a}$ and $\overset{\frown}{b}$ are equal if and only if the component forms of their origin-vectors are identical; that is, $\overset{\frown}{a} = \overset{\frown}{b}$ if and only if $[a_1,a_2,a_3] = [b_1,b_2,b_3]$, and $[a_1,a_2,a_3] = [b_1,b_2,b_3]$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Several theorems in Chapter 3 were proved to hold in the plane using components. We shall restate them here with modifications appropriate to their interpretation in space. We suggest proofs for some and leave the rest as exercises.

THEOREM 8-2. If $\vec{A} = [a_1, a_2, a_3]$ and $\vec{B} = [b_1, b_2, b_3]$,

$$\vec{A} + \vec{B} = [a_1 + b_1 , a_2 + b_2 , a_3 + b_3] .$$

We note that if the sum is $\vec{X}$, then $\overline{OX}$ and $\overline{AB}$ bisect each other at $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}\right)$. Thus

$X = (a_1 + b_1 , a_2 + b_2 , a_3 + b_3)$, and $\vec{X} = \vec{A} + \vec{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.

THEOREM 8-3. Multiplication of a vector $\vec{A}$ by a scalar r is given by

$r\vec{A} = [ra_1, ra_2, ra_3]$.

The proof is left as an exercise.

THEOREM 8-4. The inner product of two vectors $\vec{A}$ and $\vec{B}$ is given by

$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$.

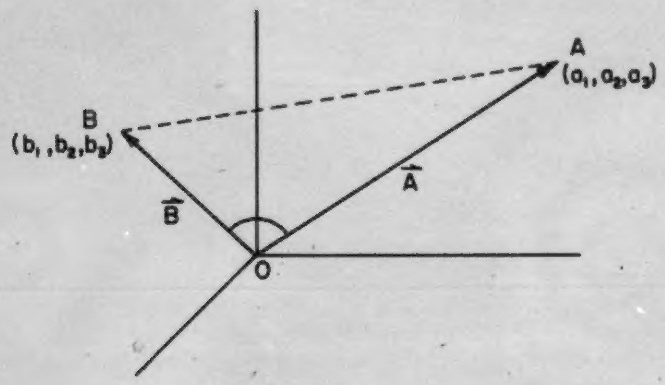


Figure 8-6

By definition $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta$; in triangle AOB we see by the Law of Cosines that

$$\cos \theta = \frac{|\vec{A}|^2 + |\vec{B}|^2 - (d(A,B))^2}{2|\vec{A}|\,|\vec{B}|}$$

Thus,

$$\overrightarrow{A} \cdot \overrightarrow{B} = |\overrightarrow{A}|\,|\overrightarrow{B}|\frac{|\overrightarrow{A}|^2 + |\overrightarrow{B}|^2 - (d(A,B))^2}{2|\overrightarrow{A}|\,|\overrightarrow{B}|}$$

$$= \frac{1}{2}(a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - ((a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2)$$

$$= \frac{1}{2}(2a_1b_1 + 2a_2b_2 + 2a_3b_3) .$$

$$= a_1b_1 + a_2b_2 + a_3b_3 .$$

THEOREM 8-5. If $\overrightarrow{X}$, $\overrightarrow{Y}$, and $\overrightarrow{Z}$ are any vectors, then

(a) $\overrightarrow{X} \cdot (\overrightarrow{Y} + \overrightarrow{Z}) = \overrightarrow{X} \cdot \overrightarrow{Y} + \overrightarrow{X} \cdot \overrightarrow{Z}$

(b) $(t\overrightarrow{X}) \cdot \overrightarrow{Y} = t(\overrightarrow{X} \cdot \overrightarrow{Y})$

Corollary. $\overrightarrow{X} \cdot (a\overrightarrow{Y} + b\overrightarrow{Z}) = a(\overrightarrow{X} \cdot \overrightarrow{Y}) + b(\overrightarrow{X} \cdot \overrightarrow{Z})$.

The proofs are left as exercises. The other theorems of Chapter 3 were not proved using components and involve no more than two vectors; hence, they apply in 3-space.

Example 2. Find the angle formed by the origin-vectors to the points $A = (2,-3,3)$ and $B = (-1,3,1)$.

Solution. We recognize that the inner product,

$$\overrightarrow{A} \cdot \overrightarrow{B} = |\overrightarrow{A}|\,|\overrightarrow{B}| \cos \theta ,$$

will help here. Since $\overrightarrow{A} = [2,-3,3]$ and $\overrightarrow{B} = [-1,3,1]$, we have

$$2 \cdot (-1) + (-3) \cdot 3 + 3 \cdot 1 = \sqrt{2^2 + (-3)^2 + 3^2}\,\sqrt{(-1)^2 + 3^2 + 1^2} \cos \theta ,$$

$$-8 = \sqrt{22} \cdot \sqrt{11} \cos \theta ,$$

and

$$\cos \theta = \frac{-4\sqrt{2}}{11}$$

$$\approx -.514 .$$

Hence

$$\theta \approx 121^\circ .$$

We recall that any vector expressed in component form in the plane may be resolved into component vectors along the axes. The component vectors in turn may be expressed as scalar multiples of unit vectors. Thus we may resolve a vector $\vec{A}$ as follows:

$$\begin{aligned}\vec{A} &= [a_1,a_2,a_3]\\ &= [a_1,0,0] + [0,a_2,0] + [0,0,a_3]\\ &= a_1[1,0,0] + a_2[0,1,0] + a_3[0,0,1]\ .\end{aligned}$$

It is customary to denote the unit vectors $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ by i , j , and k respectively. Since any vector $\vec{A}$ may be expressed as a linear combination of i , j , and k as

$$\vec{A} = a_1 i + a_2 j + a_3 k\ .$$

we say that i , j , and k form a basis for 3-space.

The use of vectors gives a concise way of describing a line in 3-space. Let (ℓ,m,n) be a triple of direction numbers of a given line L which passes through the point $P_0(x_0,y_0,z_0)$. Thus a parametric representation of L is

$$\begin{aligned}x &= x_0 + \ell t\\ y &= y_0 + mt\\ z &= z_0 + nt\ .\end{aligned}$$

The vector $\vec{D} = [\ell,m,n]$ lies on the line L' , which has a parametric representation

$$\begin{aligned}x &= \ell t\\ y &= mt\\ z &= nt\ ,\end{aligned}$$

and which is parallel to L . Thus a triple of direction numbers (ℓ,m,n) of a line L determines a vector parallel to L . Furthermore, the point $P(x,y,z)$ lies on L if and only if

$$\vec{P} = \vec{P_0} + t\vec{D}\ .$$

If L is the line which passes through two distinct points $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$, then, from Chapter 2, $(x_1 - x_0 , y_1 - y_0 , z_1 - z_0)$ is a triple of direction numbers of L. As we have just seen, this triple of direction numbers determines a vector $\vec{D}$ which is parallel to L. But

$$\vec{D} = [x_1,y_1,z_1] - [x_0,y_0,z_0] = \vec{P}_1 - \vec{P}_0 .$$

Thus, $\vec{P}_1 - \vec{P}_0$ is a vector parallel to the line through P_0 and P_1.

Example 3. Find a vector representation for the line $\overleftrightarrow{P_0P_1}$, where $\vec{P}_0 = 3i + 2j - 4k$ and $\vec{P}_1 = -2i + j + 2k$.

Solution. $P_0 = (3,2,-4)$ and $P_1 = (-2,1,2)$. Hence $\overleftrightarrow{P_0P_1}$ has $(5,1,-6)$ as a triple of direction numbers; $\vec{D} = [5,1,-6]$ is a direction vector for the line. Hence, the vector representation of the line,

$$\vec{P} = \vec{P}_0 + t\vec{D} ,$$

becomes

$$\begin{aligned} \vec{P} &= [3,2,-4] + t[5,1,-6] \\ &= [3 + 5t , 2 + t , -4 - 6t] , \end{aligned}$$

or

$$\vec{P} = (5t + 3)i + (t + 2)j - (6t + 4)k .$$

Exercises 8-5

1. Let $i = [1,0,0]$, $j = [0,1,0]$, and $k = [0,0,1]$. Find

   (a) $i \cdot j$
   (b) $i \cdot k$
   (c) $j \cdot k$
   (d) $i \cdot i$
   (e) $j \cdot j$
   (f) $k \cdot k$
   (g) $(4j + 2k) \cdot 5i$
   (h) $(3i + 2j - k) \cdot (2i + j + k)$

2. Find the cosine of the angle between the two vectors in each part of Exercise 2.

3. Given $\vec{B} = 2i + 2j - k$. Find r such that $|r\vec{B}| = 1$.

4. Let $\vec{A} = [2,3,-1]$, $\vec{B} = [3,-2,1]$, $\vec{C} = [-1,3,-2]$. Find

   (a) $2\vec{A} + 3\vec{B} - \vec{C}$
   (b) $\vec{A} - 2\vec{B} + 3\vec{C}$
   (c) $2(\vec{A} + \vec{B}) - 3(\vec{B} - \vec{C})$
   (d) $5(\vec{A} - \vec{C}) + 3(\vec{C} - \vec{A})$
   (e) $3(\vec{A} + \vec{B} - \vec{C}) + 2(\vec{A} - \vec{B} + \vec{C})$
   (f) $5(\vec{C} - \vec{A} + \vec{B}) - 3(\vec{B} + \vec{A} - \vec{C})$

5. Use values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find X so that

   (a) $\vec{A} + \vec{B} = \vec{C} + \vec{X}$
   (b) $2\vec{A} + 3\vec{B} = 4\vec{C} + 5\vec{X}$
   (c) $2(\vec{A} - \vec{B}) = 3(\vec{C} - \vec{X})$
   (d) $\vec{A} + 2\vec{X} = \vec{B} + \vec{C} - \vec{X}$
   (e) $3(\vec{X} + \vec{B}) = 2(\vec{X} - \vec{C})$
   (f) $\vec{X} + 2(\vec{X} + \vec{A}) + 3(\vec{X} + \vec{B}) = 0$

6. Use the values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find

   (a) $\vec{A} \cdot \vec{B}$
   (b) $2\vec{A} \cdot 3\vec{B}$
   (c) $3\vec{A} \cdot (\vec{B} + \vec{C})$
   (d) $2\vec{B} \cdot (3\vec{A} + 2\vec{C})$
   (e) $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$
   (f) $(2\vec{B} + 3\vec{C}) \cdot (2\vec{B} - 3\vec{C})$
   (g) $(3\vec{A} + 5\vec{B}) \cdot (3\vec{B} - 2\vec{C})$
   (h) $(\vec{A} + \vec{B} - \vec{C}) \cdot (\vec{B} - \vec{A} + \vec{C})$
   (i) $(2\vec{A} - 3\vec{B} + 4\vec{C}) \cdot (5\vec{A} - 2\vec{C} + 4\vec{B})$
   (j) $\vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C}$

7. Discuss and relate

$$\vec{A} \cdot \vec{A} \,,\ |\vec{A}|^2 \,;\ |\vec{A}|^3 \,,\ \vec{A} \cdot \vec{A} \cdot \vec{A} \,.$$

8. Given $\vec{P} = ai + bj + ck$. Give algebraic and geometric interpretations of $\dfrac{\vec{P}}{|\vec{P}|}$.

9. If $\vec{A} = 2i + 3j + 4k$ and $\vec{B} = xi - j + 3k$. Find x such that AOB is a right triangle.

10. Given $\vec{A} = 2i + 3j + 4k$ and $\vec{B} = i + j - k$, find the length of the projection of $\vec{A}$ upon $\vec{B}$.

11. Show that the line joining the end points of the vectors $\vec{A} = 2i + 3j + 4k$ and $\vec{B} = i - j + 4k$ is parallel to the xy-plane.

12. If $\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$, prove that $\vec{c} \perp (\vec{a} + \vec{b})$.

13. Describe in terms of components all unit vectors perpendicular to the xy-plane.

14. Find a vector $\perp$ to both $\vec{A} = 2i + 3j + 4k$ and $\vec{B} = i + j - k$.
    Note: There are many solutions. Can you find a general solution?

15. Find the measures of the angles of the triangle with vertices at $A = (2,-1,1)$, $B = (1,-3,5)$, $C = (3,-4,-4)$.

16. Find vector representations of the lines passing through $P = (a,b,c) \neq (0,0,0)$ which are perpendicular to $\vec{P}$.

17. Prove Theorem 8-3.

18. Prove Theorem 8-5 and its Corollary.

8-6. Vector Representations of Planes and Other Sets of Points.

In the first course in geometry plane is an undefined term; its use is described in the postulates. From the postulates we learn that a plane is a set of points and is uniquely determined by three noncollinear points. Further, if two points lie in a plane, then every point of the line containing these points also lies in the plane, and if two different planes intersect, their intersection is a line. A line and a plane were defined to be perpendicular if and only if they intersect and every line lying in the plane and passing through the point of intersection is perpendicular to the given line.

In Section 8-4 we used the fact that in space the locus of points equidistant from two given points is a plane. This led to analytic representations for planes in rectangular coordinates. In this section we shall consider another description of a plane as a locus and develop vector representations for planes.

We let M be a plane and N be a line perpendicular to M at a point P_0. Any other point P, in M, and P_0 determine a line in M, which by definition is perpendicular to N. By a theorem from geometry, every line perpendicular to N at P_0 is contained in M. Thus, we may consider M to be the locus of lines perpendicular to N at P_0. We call N a normal line to the plane.

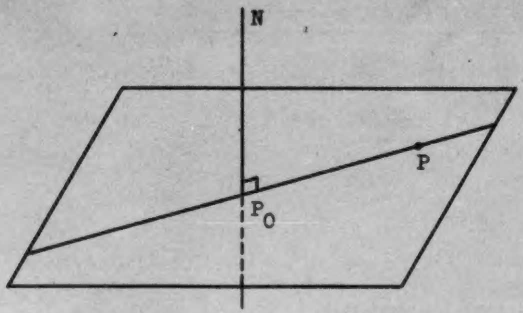


Figure 8-7

The description in terms of perpendicularity suggests a vector representation in terms of the inner product, for if $\vec{m}$ is a vector with representatives in M, and $\vec{n}$ is a vector with representatives on N, we have $\vec{m} \cdot \vec{n} = 0$. This will be clearer if we interpret the statement with origin-vectors. The vector $\vec{m}$ has a representative $\vec{m}_0$ emanating from P_0 which also lies in M. The vector $\vec{n}$ also has a representative $\vec{n}_0$ emanating from P_0 which lies on N. Hence $\vec{m}_0$ and $\vec{n}_0$ are perpendicular. Their corresponding origin-vectors $\vec{M}$ and $\vec{N}$ are perpendicular and $\vec{M} \cdot \vec{N} = 0$. By the Origin-Vector Principle we may interpret this as $\vec{m} \cdot \vec{n} = 0$.

To obtain a vector representation of the plane M, we note that if P_1 is a fixed point in M and P is any other point in M, then $\vec{P} - \vec{P}_1$ is parallel to M. Thus, we may describe the plane M as

$$\{P : (\vec{P} - \vec{P}_1) \cdot \vec{n} = 0\} .$$

We note that P_1 is also in the set.

We recall that it is possible to characterize a line which does not contain the origin in 2-space as the set of points which is perpendicular, or normal, to a directed segment $\overrightarrow{OP}$ at P. In 3-space we may describe a plane as the set of points which is normal to a directed segment $\overrightarrow{ON}$, or origin-vector $\vec{N}$, at $N \cdot \vec{N}$ is called the *normal vector* of M. If the given point of M is N, then

$$M = \{P : (\vec{P} - \vec{N}) \cdot \vec{N} = 0\} .$$

If we let $P = (x,y,z)$, $|\overrightarrow{N}| = p$, and (λ,μ,ν) be the triple of direction cosines of $\overrightarrow{ON}$, we have

$$\overrightarrow{P} = [x,y,z] \ ,$$

$$N = (\lambda p\, , \mu p\, , \nu p) \ ,$$

and

$$\overrightarrow{N} = [\lambda p\, , \mu p\, , \nu p] = p[\lambda,\mu,\nu] \ .$$

Thus

$$(\overrightarrow{P} - \overrightarrow{N}) \cdot \overrightarrow{N} = ([x,y,z] - p[\lambda,\mu,\nu]) \cdot p[\lambda,\mu,\nu] = 0 \ ,$$

which, since $p \neq 0$, is equivalent to

$$[x,y,z] \cdot [\lambda,\mu,\nu] - p[\lambda,\mu,\nu] \cdot [\lambda,\mu,\nu] = 0 \ ,$$

or

$$\lambda x + \mu y + \nu z - p(\lambda^2 + \mu^2 + \nu^2) = 0$$

Since $\lambda^2 + \mu^2 + \nu^2 = 1$, we have

$$M = \{(x,y,z) : \lambda x + \mu y + \nu z - p = 0\} \ ,$$

an analytic representation of the plane in terms of the normal form of its equation. We note that (λ,μ,ν) are direction cosines of the normal segment and that p is the distance between the origin and the plane.

Example 1. Find an equation of the plane which is perpendicular to the vector $A = [6,-4,3]$ at the point A .

Solution. We have

$$([x,y,z] - [6,-4,3]) \cdot [6,-4,3] = 0 \ ,$$

$$[x - 6\, , y + 4\, , z - 3] \cdot [6,-4,3] = 0 \ ,$$

and

$$6x - 36 - 4y - 16 + 3z - 9 = 0 \ ,$$

or

$$6x - 4y + 3z - 61 = 0 \ .$$

Again we note that the coefficients are direction numbers of normal lines to the plane.

Example 2. Show that if $P_0 = (x_0,y_0,z_0)$ and $P_1 = (x_1,y_1,z_1)$ are two distinct points in a plane with equation $ax + by + cz + d = 0$, then every point of $\overleftrightarrow{P_0P_1}$ is in the plane.

Solution. Any point $P = (x,y,z)$ on line has the parametric representation

$$x = x_0 + (x_1 - x_0)t$$
$$y = y_0 + (y_1 - y_0)t$$
$$z = z_0 + (z_1 - z_0)t ,$$

and is in the plane if its coordinates satisfy the equation $ax + by + cz + d = 0$. The left member becomes

$$\begin{aligned} &a(x_0 + (x_1 - x_0)t) + b(y_0 + (y_1 - y_0)t) + c(z_0 + (z_1 - z_0)t) + d \\ &= (ax_0 + by_0 + cz_0 + d) + (ax_1 + by_1 + cz_1)t - (ax_0 + by_0 + cz_0)t \\ &= 0 + (-d)t - (-d)t = 0 . \end{aligned}$$

Therefore, any point of the line is contained in the plane.

We may use vectors, as we did in Section 3-6, to describe other sets of points in space.

Example 3. Find a vector representation for the line segment determined by the vectors $\widehat{A} = [2,-1,3]$ and $\widehat{B} = [-1,4,7]$ in terms of a single parameter p .

Solution. From the development above, $\overline{AB} = \{X : \widehat{X} = p\widehat{A} + q\widehat{B}$ where $p \geq 0$, $q \geq 0$, and $p + q = 1\}$.

Since $p + q = 1$, $q = 1 - p$; since $q \geq 0$, $1 - p \geq 0$ or $p \leq 1$. Since $p > 0$, the combined restriction on p is that $0 \leq p \leq 1$. By substitution,

$$\begin{aligned} p\widehat{A} + q\widehat{B} &= p[2,-1,3] + (1 - p)[-1,4,7] \quad \text{where } 0 \leq p \leq 1 \\ &= [2p,-p,3p] + [p - 1 , 4 - 4p , 7 - 7p] \quad \text{where } 0 \leq p \leq 1 . \\ &= [3p - 1 , 4 - 5p , 7 - 4p] \quad \text{where } 0 \leq p \leq 1 . \end{aligned}$$

and

$$\overline{AB} = \{X : \widehat{X} = [3p - 1 , 4 - 5p , 7 - 4p] , \text{ where } 0 \leq p \leq 1\}$$

Example 4. Find a vector representation of the point which divides the directed segment $\overset{\frown}{AB}$ in the ratio $\frac{1}{2}$.

Solution.

$$\begin{aligned}\vec{X} &= \frac{2}{1+2}\vec{A} + \frac{1}{1+2}\vec{B}\\ &= \frac{2}{3}[2,-1,3] + \frac{1}{3}[-1,4,7]\\ &= [\frac{4}{3},-\frac{2}{3},2] + [-\frac{1}{3},\frac{4}{3},\frac{7}{3}]\\ &= [1,\frac{2}{3},\frac{13}{3}]\end{aligned}$$

Alternatively, if we think of the parameter as a coordinate of the point, then for the desired point $p = \frac{2}{3}$. Substituting this value in the expression obtained in Example 3, we obtain

$$\begin{aligned}\vec{X} &= [3\cdot\frac{2}{3} - 1 , 4 - 5\cdot\frac{2}{3} , 7 - 4\cdot\frac{2}{3}]\\ &= [1,\frac{2}{3},\frac{13}{3}]\end{aligned}$$

Example 5. Find a vector representation for the ray opposite to $\overrightarrow{BA}$ in terms of a single parameter q.

Solution. The ray opposite to $\overrightarrow{BA} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p \leq 0$ and $p + q = 1\}$. Since

$$p = 1 - q \leq 0 ,$$

therefore

$$q \geq 1 .$$

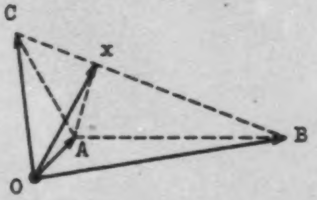
$$\begin{aligned}p\vec{A} + q\vec{B} &= (1 - q)[2,-1,3] + q[-1,4,7] \quad \text{where} \quad q \geq 1\\ &= [2 - 2q , q - 1 , 3 - 3q] + [-q,4q,7q] \quad \text{where} \quad q \geq 1\\ &= [2 - 3q , 5q - 1 , 3 + 4q] \quad \text{where} \quad q \geq 1 .\end{aligned}$$

The ray opposite to $\overrightarrow{BA} = \{X : \vec{X} = [2 - 3q , 5q - 1 , 3 + 4q]$, where $q \geq 1\}$.

Example 4. Suppose $\vec{A}$, $\vec{B}$, and $\vec{C}$ are the vectors whose terminal points are the vertices of a triangle. Can we represent the triangular region, the interior of the triangle, and the triangle itself, in terms of these vectors and two parameters?

Solution. We write $\overline{BC}$ as $\{X : \vec{X} = q\vec{B} + (1 - q)\vec{C}$ where $0 \le q \le 1\}$ as in Example 3 above.

Now the triangular region is the union of the segments $\overline{AX}$ or



$$\{Y : \vec{Y} = p\vec{A} + (1 - p)\vec{X} \text{ where } 0 \le p \le 1\}$$
$$= \{Y : \vec{Y} = p\vec{A} + (1 - p)[q\vec{B} + (1 - q)\vec{C}] \text{ where } 0 \le p \le 1 \text{ and } 0 \le q \le 1\}$$
$$= \{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)(1 - q)\vec{C} \text{ where } 0 \le p \le 1 \text{ and } 0 \le q \le 1\} .$$

The interior of the triangle ABC will be

$$\{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)\vec{C} \text{ where } 0 < p < 1 \text{ and } 0 < q < 1\} .$$

The triangle is

$$\{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)(1 - q)\vec{C} \text{ where } (p = 0 \text{ and } 0 \le q \le 1) \text{ or } (q = 0 \text{ and } 0 \le p \le 1) \text{ or } (q = 1 \text{ and } 0 \le p \le 1)\} .$$

(We can write these results more neatly if we let $r = (1 - p)q$ and $s = (1 - p)(1 - q)$. Then $p + r + s = 1$ and the triangular region is

$$\{Y : \vec{Y} = p\vec{A} + r\vec{B} + s\vec{C} \text{ where } p\text{ , } r\text{ , and } s \text{ are non-negative and } p + r + s = 1\}$$

This form is easier to recall.)

Exercises 8-6

1. Find an equation of the plane which has $[7,-3,5]$ as a normal vector and which contains the point $(0,0,3)$.

2. Find an equation of the plane with the normal vector

   (a) $[2,-3,1]$
   (b) $[-2,4,-7]$
   (c) $[3,-5,4]$
   (d) $[-1,-1,6]$

3. Find the distance from $(0,0,0)$ to the plane

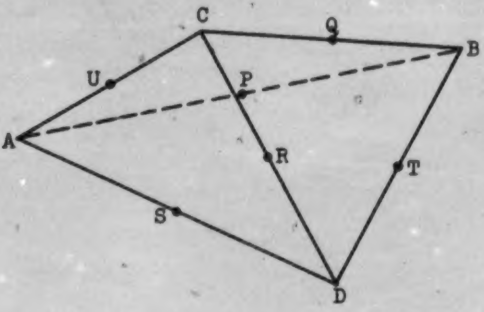
   (a) $2x + 3y - z = 5$
   (b) $5x - 3y + 2z = 8$
   (c) $ax + by + cz = d$

4. In the figure below, consider ABCD to be a 3-dimensional figure. (This is known as a tetrahedron and has 4 faces and 6 edges.)

   (a) Show that the lines through the midpoints of opposite edges are concurrent.

   (b) Show that PTRU and QUST are parallelograms.

   (c) Show that the point of concurrency is the midpoint of each segment.



5. Show that if $P_1 = (x_1, y_1, z_1)$ and $M = \{(x,y,z) : \lambda x + \mu y + \nu z - p = 0\}$, then the distance between P_1 and M is

$$|\lambda x_1 + \mu y_1 + \nu z_1 - p| .$$

6. Find vector representations, in terms of a single parameter, for the sets described below.

   (a) $\overleftrightarrow{AB}$ where $\vec{A} = [4,-7,5]$ and $\vec{B} = [4,2,3]$

   (b) $\overline{AB}$ where $\vec{A} = [3,4,2]$ and $\vec{B} = [-2,3,3]$

   (c) $\overrightarrow{AB}$ where $\vec{A} = [3,4,2]$ and $\vec{B} = [-2,3,3]$

   (d) $\overrightarrow{BA}$ where $\vec{A} = [3,4,2]$ and $\vec{B} = [-2,3,3]$

7. Find the vector representations of the midpoints and trisection points of the following line segments:

   (a) $\overline{AB}$ where $\vec{A} = [0,0,0]$ and $\vec{B} = [6,12,15]$

   (b) $\overline{AB}$ where $\vec{A} = [-3,2,7]$ and $\vec{B} = [10,-11,19]$

   (c) $\overline{AB}$ where $\vec{A} = [a_1,a_2,a_3]$ and $\vec{B} = [b_1,b_2.b_3]$

8. Find the vector representations of the points which divide the directed segment $\overrightarrow{PQ}$ in the ratio $\frac{r}{s}$ where:

   (a) $\vec{P} = [-3,-2,-1]$, $\vec{Q} = [3,2,1]$, and $\frac{r}{s} = 1$

   (b) $\vec{P} = [-1,4,-8]$, $\vec{Q} = [9,-5,7]$, and $\frac{r}{s} = \frac{1}{5}$

   (c) $\vec{R} = [2,3,1]$, $\vec{S} = [1,-2,4]$, and $\frac{r}{s} = \frac{3}{1}$

9. Given the triangle ABC with $\vec{A} = [2,3,1]$, $\vec{B} = [-1,2,4]$, and $\vec{C} = [1,4,-2]$.

   (a) Describe the triangular region, its interior, and the triangle itself, using these vectors and two parameters.

   (b) Show that $[1,3,1]$ is a vector whose terminal point is an interior point of the triangle.

   (c) Show that $[-4,-5,-6]$ is a vector whose terminal point is an exterior point of the triangle.

Challenge Problem

1. Given the four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$, whose terminal points are not coplanar, find an expression for the tetrahedral region ABCD in terms of these vectors and three parameters.

8-7. Summary.

We have extended the rectangular coordinate system to 3-space and have considered the analytic and vector representations of lines and planes in 3-space. In Chapter 9 we shall consider the representation and sketching of other curves and surfaces. We shall also consider two extensions of polar coordinates to 3-space.

We have also suggested that we may interpret algebraic relationships in four variables in a 4-space, which may be helpful even though we cannot visualize it. The extension is, of course, possible to spaces of more dimensions. We are in a position to make several conjectures based on our observations in 2-space and 3-space. In 2-space the general linear equation in 2 variables describes a line, a one-dimensional figure; in 3-space the general linear equation in 3 variables describes a plane, a 2-dimensional figure. Thus, in n-space we might expect the general linear equation in n-variables to describe a figure with n-1 dimensions.

In 2-space we are able to describe a line either by a linear equation or by a parametric representation in one parameter; in 3-space we still have the parametric representation of a line in one parameter, but the alternative is the common solution of two linear equations, which is awkward. Some of the later exercises show that we may also describe regions in a plane by a parametric representation in two parameters. Our conjecture might be that in spaces with enough dimensions we may describe one-dimensional figures with parametric representations in one parameter, 2 dimensional figures with parametric representations in two parameters, and, in general, n-dimensional figures with parametric representations in n parameters.

Review Exercises

In Exercises 1 to 8, write an equation of the locus of a point which satisfies the stated conditions.

1. A point 5 units above the xy-plane.
2. A point 5 units from the yz-plane.
3. A point equidistant from the xy- and the yz-planes.
4. A point 2 units from the x-axis.
5. A point a units from the origin.
6. A point r units from the point $(2,-1,0)$.
7. A point equidistant from the point $(1,2,3)$ and the plane with equation $z = 2$.
8. A point that lies in the plane determined by the points $(3,1,2)$, $(1,2,3)$, $(2,2,2)$.

Sketch the graph of the equations in Exercises 9 to 14.

9. $x + y - 4 = 0$
10. $2z - 7 = 0$
11. $4x + 9y - 6z + 36 = 0$
12. $x - y + z + 3 = 0$
13. $x = 5 - 3t,\ y = 2 + t,\ z = 3 - 4t$
14. $\frac{x-5}{-3} = \frac{y-2}{-2} = \frac{z-3}{4}$

In Exercises 15-20, graph and describe the geometric representation in one-space and 2-space, and discuss a possible meaning in 3-space.

15. $\{x : x - 3 = 0\}$

16. $\{x : -1 < x < 3\}$

17. $\{x : |x| - 3 = 0\}$

18. $\{x : |x| \geq 3\}$

19. $\{x : |x| \leq 5\}$

20. $\{x : x(x - 1)(x + 2) = 0\}$

21. Graph and describe R_1, R_2, and R_3 for one space, 2-space, and 3-space if

$R_1 = \{(x,y) : |x| < 2\}$, $R_2 = \{(x,y) : |y| < 2\}$, $R_3 = R_1 \cap R_2$.

22. Discuss Exercise 21 if $<$ is changed to $\leq$. What geometric interpretation can you give for $R_1 \cup R_2$?

23. Graph and describe $\{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$. What is the graph if $\leq$ is changed to $<$?

In Exercises 24 to 26, use the four points: $A(-2,1,3)$, $B(3,1,-2)$, $C(2,3,-1)$, $D(1,-3,2)$, and the four planes:

$M_1 : 2x - 3y + z + 4 = 0$, $M_2 : 3x - y + 2z - 3 = 0$, $M_3 : x + 2y - 3z + 2 = 0$

$M_4 : -x + y + z - 1 = 0$.

24. Find the distance from each of the points A , B , C , D , O to each of the planes:

(a) M_1

(b) M_2

(c) M_3

(d) M_4

25. Find, in symmetric form, equations of the lines determined by:

(a) $\{M_1,M_2\}$

(b) $\{M_1,M_3\}$

(c) $\{M_1,M_4\}$

(d) $\{M_2,M_3\}$

(e) $\{M_2,M_4\}$

(f) $\{M_3,M_4\}$

26. Find parametric equations for each of the lines referred to in Problem 25.

27. Show that the space quadrilateral ABCD , where $A = (-2,3,2)$, $B = (-4,5,8)$, $C = (1,1,4)$, $D = (3,-1,-2)$, is a parallelogram.

28. Show that the medians of triangle ABC, where $A = (0,0,0)$, $B = (2,4,6)$, $C = (-4,2,-8)$, are concurrent.

29. For what value of a are the points $(3,2,3)$, $(1,-4,2)$, $(2,14,5)$ collinear?

30. If $(2,1,4)$, $(0,4,-2)$, $(a,-2,-4)$ are the vertices of a triangle with a right angle at vertex $(0,4,-2)$, find a.

Chapter 9

QUADRIC SURFACES

9-1. What Is a Quadric Surface?

If you know what is meant by "quadratic equation," you might guess what is meant by "quadric surface". The locus, if one exists, of an equation of the second degree in rectangular coordinates for 3-space is called a quadric surface. Each of these surfaces has an important property: all plane sections are conics. There are many surfaces other than quadric surfaces, and there are more quadric surfaces than the ones we shall introduce. We shall limit our discussion to the most useful and easily recognized ones. You will recognize spheres, cones, and cylinders. Some of the other surfaces may be less familiar to you, but, inasmuch as all intersections of these surfaces with planes are conic sections, you should have little difficulty visualizing even those quadric surfaces which are new to you.

When we apply mathematics to physical problems, we find that a drawing which depicts the physical relations in the problem can be useful. Our principal aim in this chapter is to develop methods for visualizing surfaces and curves in 3-space. Such configurations frequently occur in science and calculus courses. We shall give directions involving only simple figures and equations, but the methods are general and can be extended to more complicated cases. We also shall indicate how equations representing quadric surfaces or space curves may be simplified.

Some ability in the sketching of geometric figures is required in this chapter; you must make drawings of three-dimensional objects on a two-dimensional surface. Also, we shall rely heavily upon the material which you learned in Chapters 5, 6, and 7.

9-2. Spheres and Ellipsoids.

You are familiar with the graph of the points in a plane at a given distance from a given point, and you also know an equation of this graph. If the given point is taken as the origin and the given distance is 4 , the equation is

$$x^2 + y^2 = 16 .$$

Now suppose we consider this same problem in 3-space. You know that the locus is a sphere of radius 4 , but let us proceed as we would if you did not know this. We shall use various methods to "discover" the shape of this familiar surface. Later you will use the same methods to find the shape of unfamiliar surfaces.

A sphere is defined as the set of points each of which is at a given distance from a given point. It always will be possible to select this given point (the center) as the origin of a rectangular coordinate system. Such a choice will simplify the algebraic representation of the sphere.

We wish to examine the set of points, each of which is a distance 4 from the origin, $0 = (0,0,0)$. For each such point $P = (x,y,z)$, the condition is

$$\sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = 4$$

or (1) $$x^2 + y^2 + z^2 = 16 .$$

An attempt to visualize this sphere by plotting points, such as $(2,3,\sqrt{3})$, $(1,\sqrt{6},3)$, $(\sqrt{2},-3,\sqrt{5})$, not only is tedious but, even when a great many points have been plotted, does not reveal the sphere we expect.

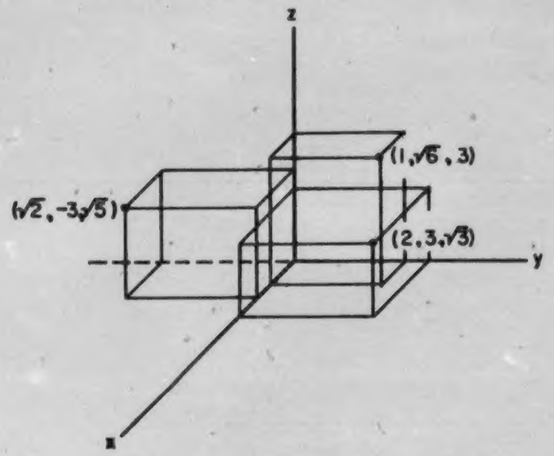


Figure 9-1

It is more illuminating to exploit the similarity between the equation of a sphere and the equation of a circle. For instance, the equation

$$y^2 + z^2 = 16 \tag{2}$$

not only closely resembles our equation (1) of the sphere under discussion, but Equation (2) represents a part of this sphere. It represents, of course, the intersection of the sphere and the yz-plane $(x = 0)$ shown in Figure 9-2. The intersection of a quadric surface and a coordinate plane is called a trace.

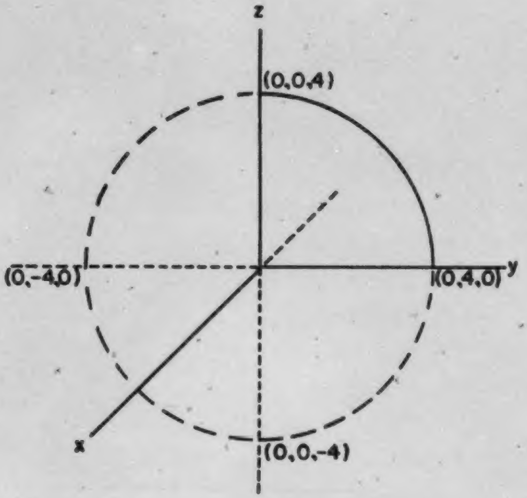


Figure 9-2

The algebraic representation of this trace if the simultaneous solution of Equation (1) and $x = 0$. The traces in the other coordinate planes are found by taking $y = 0$ and $z = 0$. We show in the figure only those parts of traces which are in the boundaries of the first octant.

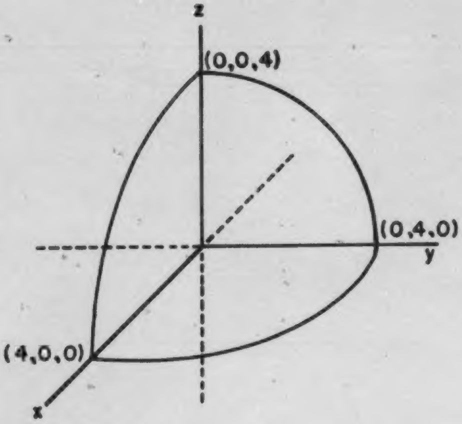


Figure 9-3

In some problems we need help in drawing the traces. In this event we locate the intercepts- the points of intersection of the surface with the coordinate axes. For Equation (1) the values are 4 and -4 on each axis.

Once the traces are indicated, as in Figure 9-3, we begin to see the shape of the surface. Next we investigate the shape of the rest of the surface by slicing it and looking at each slice. Such slices are called sections; they are the curves formed by the surface and planes cutting it. The traces, of course, are special cases of sections. Let us make our slices parallel to the xy-plane. An equation of the parallel plane one unit above the xy-plane is $z = 1$; we substitute for z in Equation (1), which becomes

$$x^2 + y^2 + 1 = 16 ,$$

or

$$x^2 + y^2 = 15 .$$

We see that this is an equation of a circle in a plane parallel to the xy-plane, with radius $\sqrt{15} \approx 3.9$, and with its center on the z-axis; we add to the figure, in the plane $z = 1$, the part of the circle in the first octant. We continue in this fashion, letting z assume the values 2 and 3. Each section is a circle, and the radii are approximately 3.5 and 2.6, respectively. We have added parts of these circles in Figure 9-4. When $z = 4$ we have

$$x^2 + y^2 = 0 ,$$

which represents the point $(0,0,4)$. For any value of z larger than 4, there is no locus.

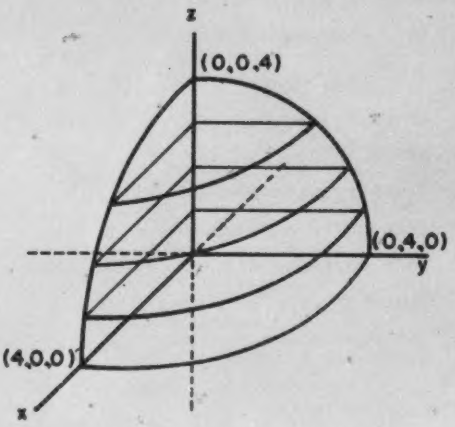


Figure 9-4

Now we consider sections parallel to the yz-plane, giving the same numerical values to x that we gave to z. Again we find that the sections are circles, which we may add to our drawing (Figure 9-5). We might also investigate sections parallel to the xz-plane if this appears to aid our visualization.

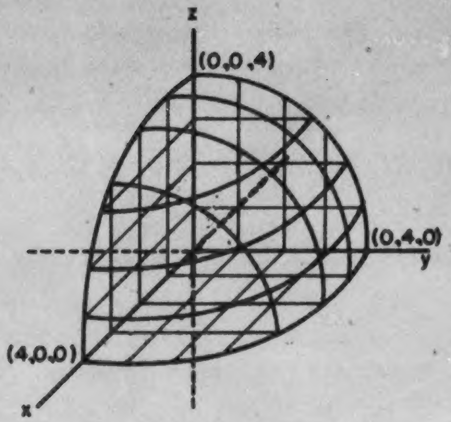


Figure 9-5

This has probably seemed a slow and labored procedure to get a drawing of such a familiar surface as the sphere, but we hope that you will now be able to apply the same methods to other equations in order to visualize and draw the surfaces they represent.

When graphing in three dimensions it is helpful, as it was in two, to investigate symmetry. The definitions of point-symmetry and line-symmetry given in Section 6-2 hold for 3-space, but a more useful idea is that of symmetry with respect to a plane. A set of points S is symmetric with respect to a fixed plane M if and only if for each point P of S there is a corresponding point P' of S such that M is the perpendicular bisector of $\overline{PP'}$. Here we shall investigate symmetry only with respect to the coordinate planes. We list the tests: a graph will be symmetric with

$$\text{respect to the } \left\{\begin{matrix}\text{xy-plane}\\ \text{yz-plane}\\ \text{xz-plane}\end{matrix}\right\} \text{ if, whenever } (x_1,y_1,z_1) \text{ is on the graph, so also is } \left\{\begin{matrix}(x_1,y_1-z_1)\\ (-x_1,y_1,z_1)\\ (x_1,-y_1,z_1)\end{matrix}\right.$$

If a surface is symmetric with respect to all three coordinate planes, it is also symmetric with respect to the origin and each axis. A sphere, of course, meets all these tests for symmetry.

When a surface is symmetric with respect to all three coordinate planes, the part of it in any octant is repeated in all the other octants. In such cases we need draw only that part in the first octant, since this makes our drawing less complicated.

The sphere we have been considering has its center at the origin; the equation for such a sphere can always be written in the form

$$x^2 + y^2 + z^2 = a^2 \tag{3}$$

where $|a|$ is the radius. Note that the terms containing x, y, z all have the coefficient 1 .

Consider the equation

$$4x^2 + y^2 + 4z^2 = 100 \; . \tag{4}$$

What quadric surface does this represent? We begin, as before, by drawing the traces. To find the trace in the yz-plane, we let $x = 0$ in Equation (4), obtaining $\frac{y^2}{100} + \frac{z^2}{25} = 1$. We recognize that this trace is an ellipse, as shown in Figure 9-6. When we let $z = 0$, we again obtain an ellipse. However, when $y = 0$, $x^2 + z^2 = 25$; the trace is a circle. Again we shall picture only those portions of the traces lying in the boundaries of the first octant. These are shown in Figure 9-7.

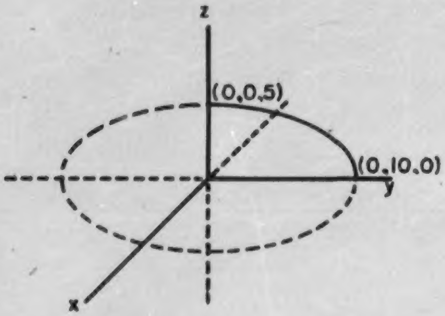


Figure 9-6

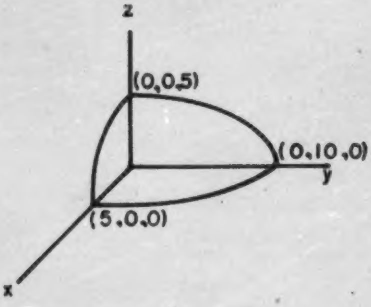


Figure 9-7

Now we find the sections as before; those parallel to the xy- and yz-planes are ellipses; the ones parallel to the xz-plane are circles. It is common practice to select just one set of sections to illuminate the drawing; if one set consists of circles, this is the usual choice. These sections are shown in Figure 9-8.

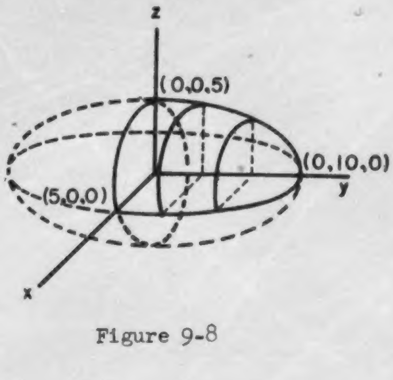


Figure 9-8

The surface we have been sketching belongs to a class called *ellipsoids*. They are so named because the sections parallel to the coordinates planes are ellipses (or circles, which may be considered special cases of ellipses). These surfaces have equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 , \tag{5}$$

where the numbers $\pm a$, $\pm b$, $\pm c$, are the x- , y- , z- intercepts respectively. The segments of the axes joining the intercept points are called *axes* of the *ellipsoid*.

If two of the axes of an ellipsoid have equal length, the surface is called a *spheroid*, because it resembles a sphere. These are of two kinds. If the third axis is longer than the others as is illustrated in Figure 9-8, the spheroid is called a *prolate spheroid* and resembles a football or a watermelon. If the third axis is shorter than the other two, the surface is called an *oblate spheroid* and appears flattened like the earth or a "Yo-Yo" top.

When $a = b = c$ in Equation (5), we have the equation of a sphere. A sphere, then, is a special kind of ellipsoid in much the same sense that a circle is a special kind of ellipse. Before we conclude this section we should ask again, "What quadric surface does Equation (4) represent"? Following what is a good general procedure, you should write Equation (4) in the form of Equation (5) and then name the surface according to the above descriptions.

Exercises 9-2

In Exercises 1 to 12, discuss and sketch the surface represented. Include intercepts, traces, and the name of the surface. Draw several of the sections parallel to one of the coordinate planes.

1. $x^2 + y^2 + z^2 = 25$
2. $4x^2 + 4y^2 + 4z^2 = 9$
3. $9x^2 + 9y^2 + 9z^2 = 0$
4. $9x^2 + 4y^2 + 9z^2 = 36$
5. $9x^2 + 9y^2 + 4z^2 = 36$
6. $4x^2 + 25y^2 + 25z^2 = 100$
7. $4x^2 + 9y^2 + 4z^2 = 36$
8. $9x^2 + 9y^2 + 25z^2 = 225$
9. $9x^2 + 25y^2 + 25z^2 = 225$
10. $4x^2 + 9y^2 + 16z^2 = 144$
11. $9x^2 + 4y^2 + 16z^2 = 144$
12. $16x^2 + 9y^2 + 4z^2 = 144$

13. Use the definition of sphere to write an equation of a sphere with center (x_0,y_0,z_0) and radius r.

14. Show that the equation you obtained in Exercise 13 can always be written in the form

$$x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0 .$$

Does every equation written in this form represent a sphere? Justify your answer.

15. Find, in the form in Exercise 14, equations of the spheres with the given center (C) and radius (r).

(a) $C = (2,1,3)$, $r = 5$

(b) $C = (0,-1,2)$, $r = 2$

(c) $C = (1,3,-2)$, $r = \sqrt{2}$

(d) $C = (\frac{1}{3}, -1, \frac{1}{2})$, $r = 1$

(e) $C = (\frac{1}{2}, \frac{1}{4}, -\frac{1}{2})$, $r = \frac{1}{2}$

(f) $C = (1.5, -.5, 2.5)$, $r = 3$

16. Determine whether the following equations represent spheres. For each sphere, give the radius and the coordinates of the center.

(a) $3x^2 + 3y^2 + 3z^2 - 9 = 0$

(b) $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

(c) $x^2 + y^2 + z^2 - 4y + 2z - 20 = 0$

(d) $x^2 + y^2 + z^2 + 6x - 8y + 14z + 72 = 0$

(e) $x^2 + y^2 + z^2 + 4x - 6y + 13 = 0$

(f) $x^2 + y^2 + z^2 - 2x + 6y + 14 = 0$

(g) $36x^2 + 36y^2 + 36z^2 - 36x - 48y + 72z + 52 = 0$

(h) $16x^2 + 16y^2 + 16z^2 - 24x - 64y + 16z + 41 = 0$

17. If $A = (1,2,3)$ and $B = (-1,0,7)$, what is an equation of the sphere that has $\overline{AB}$ as diameter?

18. Write an equation of an ellipsoid with x-, y-, and z-intercepts ± 3, ± 7, ± 5, respectively.

Challenge Problems

1. Write an equation of an ellipsoid with center at the point $(3,-1,2)$, and with axes parallel to the x-, y-, and z-axes and of lengths 12, 8, and 24 respectively.

2. Points $P = (0,3,1)$, $Q = (-2,0,2)$, $R = (1,1,4)$, and $S = (-3,3,2)$ are points of a sphere. What is an equation of the sphere? Will any four distinct points determine a sphere?

9-3. The Paraboloid and the Hyperboloid.

What is the locus of a point equidistant from a given point F and a given plane M ? We shall assume that the distance from F to M is 4 . The geometric condition for the locus is similar to the one which defines a parabola. With this in mind we let the line through F perpendicular to M be the y-axis and let the origin be the midpoint of the normal segment from F to M . Then $F = (0,2,0)$ and the equation of M is $y + 2 = 0$. The required point $P = (x,y,z)$ must meet the condition

$$\sqrt{x^2 + (y - 2)^2 + z^2} = \left|\frac{y + 2}{\sqrt{1}}\right| .$$

Squaring, we have $x^2 + y^2 - 4y + 4 + z^2 = y^2 + 4y + 4$;

hence (1) $$x^2 + z^2 = 8y$$

is an equation for the locus.

Now we must decide what the graph of this equation looks like. We shall use the same methods we applied to the equation of the sphere. If we look for intercepts, we find that the only intersection of the surface with the axes is the origin, $(0,0,0)$. The trace in the xy-plane is the parabola $x^2 = 8y$; in the yz-plane, the parabola $z^2 = 8y$. The trace in the xz-plane is the single point O , given by the equation $x^2 + z^2 = 0$. We notice that in Equation (1) y cannot have negative values; hence no part of the surface is to the left of the xz-plane.

We next investigate the sections parallel to the xz-plane. When $y = 1$, we have $x^2 + z^2 = 8$, a circle with radius $2\sqrt{2}$. For $y = 2$, we have a circle of radius 4 , and so on. Thus the surface may be thought of as formed by a succession of circles, beginning with the point-circle and with radius increasing without limit as y increases. This bullet-shaped surface (Figure 9-9) is called a paraboloid. It is also called a paraboloid of revolution, as it may be generated by revolving a parabola about its axis. The reflector usually called a parabolic reflector is really a paraboloid.

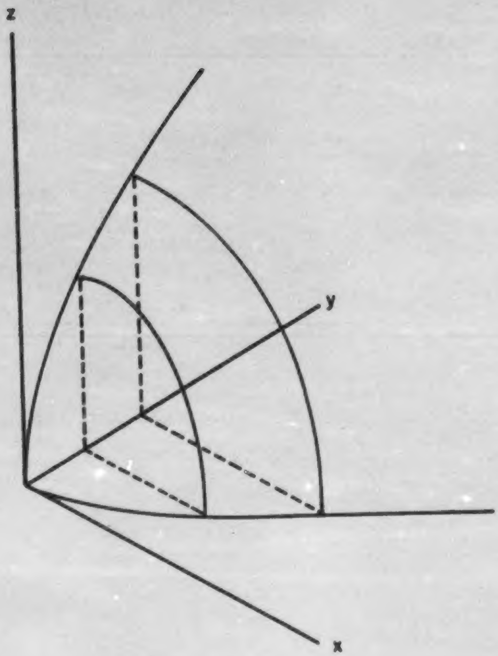


Figure 9-9

A more general equation of a parabolid is of the form

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = by \quad (2)$$

The traces of this surface in the xy- and yz-planes are parabolas, but the sections parallel to the xz-plane are ellipses or circles. This surface is called an elliptic paraboloid.

We turn now to the equation

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{25} = 1 , \quad (3)$$

and find that the x- and y-intercepts are ± 2 and ± 3 respectively, but that there are no z-intercepts. The trace in the xy-plane is an ellipse; in the other coordinate planes the traces are hyperbolas. Since ellipses are easier to draw than hyperbolas, let us make our sections parallel to the xy-plane. When $z = 1$ we have

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 + \frac{1}{25} ,$$

representing an ellipse very much like the one which is a trace in the xy-plane. We continue, finding that for numerically larger values of z the sections will be ellipses with increasingly larger intercepts. This surface (Figure 9-10) is called a hyperboloid of one sheet, or an elliptic hyperboloid. Its equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 . \tag{4}$$

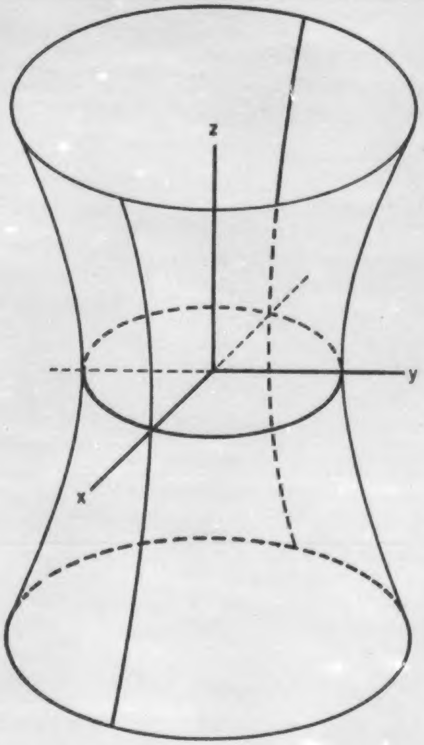


Figure 9-10

Next we consider the equation

$$-\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = 1 . \tag{5}$$

Here there are no x- or y-intercepts; the z-intercepts are ± 5 . The traces in the yz- and xz-planes are hyperbolas. Again we make our sections parallel to the xy-plane. If we write the Equation (5) in the form

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{z^2}{25} - 1 ,$$

we see that when $|z| < 5$ there are no real values of x or y .

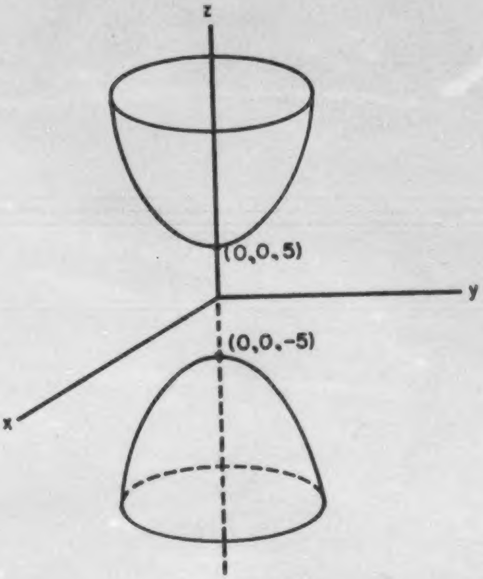


Figure 9-11

When $z = 5$ the section is the point $(0,0,5)$; for $z = -5$, we have the point $(0,0,-5)$. For $|z| > 5$ the sections are ellipses, whose axes increase as $|z|$ increases. Thus our surface may be thought of as two separate piles of ellipses. It is called a hyperboloid (or elliptic hyperboloid) of two sheets.

Exercises 9-3

Discuss and sketch the surfaces represented by the equations in Exercises 1 to 12.

1. $y^2 + z^2 = 4x$
2. $x^2 + y^2 = 16z$
3. $4x^2 + 4z^2 = 16y$
4. $4x^2 + 9z^2 = 144y$
5. $9x^2 + 4z^2 = 144y$
6. $9y^2 + 4z^2 = 144x$
7. $9x^2 + 9y^2 - z^2 = 36$
8. $9x^2 - 4y^2 + 9z^2 = 36$
9. $x^2 - 9y^2 + 4z^2 = 36$
10. $4x^2 - 25y^2 + 4z^2 = 100$
11. $4x^2 - 9y^2 + z^2 = 144$
12. $x^2 - y^2 + z^2 - 1 = 0$

13. We observed that, for the hyperboloid whose graph is given by Equation (3), the sections parallel to the xy-plane are ellipses. Prove that these ellipses have the same eccentricity.

Challenge Problems

The surfaces represented by the following equations are called *hyperbolic paraboloids*. Discuss and sketch them.

1. $4x^2 - 9y^2 = 36z$.
2. $16y^2 - 9x^2 = 144z$.
3. $y^2 - z^2 = x$.

9-4. Cylinders.

Equations of the quadric surfaces which we have investigated have contained all three variables. What if an equation contains only two variables? Suppose the equation is

$$x^2 + y^2 = 25 . \tag{1}$$

We find the x- and y-intercepts, and note that there are no z-intercepts. The trace in the xy-plane is a circle of radius 5 with the center at 0 ; in each of the other coordinate planes it is two straight lines, parallel to the coordinate axis. The sections parallel to the xy-plane are all circles of radius 5 with their centers on the z-axis. From Figure 9-12 we recognize the surface as a cylinder.

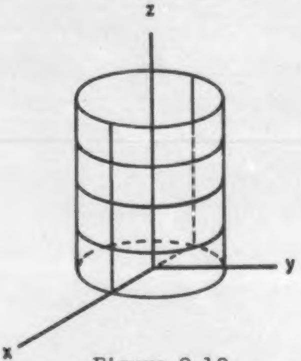


Figure 9-12

A *cylindrical surface*, or *cylinder*, is the surface formed when a line moves in space so that it always has the same direction numbers and intersects a fixed plane curve. The plane curve is called a *directrix*; the lines are called *generators* or *elements*. A part of such a surface is shown in

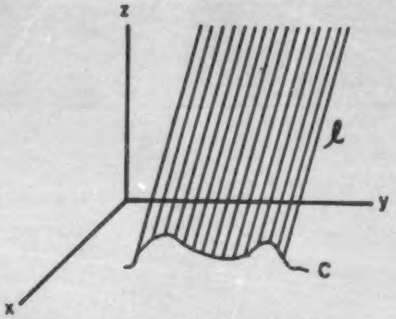


Figure 9-13

Figure 9-13; the curve c in the xy-plane is a directrix, the line ℓ an element. For the circular cyliner in Figure 9-12, any one of the circles we have drawn might be considered a directrix, and any of the lines of the cylinder an element.

We shall restrict our examples to cylinders with elements parallel to an axis. In such cases one of the variables is missing from the equation. For example, we shall consider the equation

$$\frac{x^2}{36} + \frac{z^2}{9} = 1 \,. \tag{2}$$

Let us see if we can show that this surface satisfies our definition of a cylinder. If it is a cylinder then the trace in the xz-plane, the ellipse with equations

$$\frac{x^2}{36} + \frac{z^2}{9} = 1 \,,\; y = 0 \,,$$

must be a directrix. We select any point of this ellipse, say $P = (4,0,\sqrt{5})$. We find that for any value y , the point $(4,y,\sqrt{5})$ is a point of the surface. All such points lie on the line ℓ perpendicular to the xz-plane at P ; hence ℓ is an element of the cylinder.

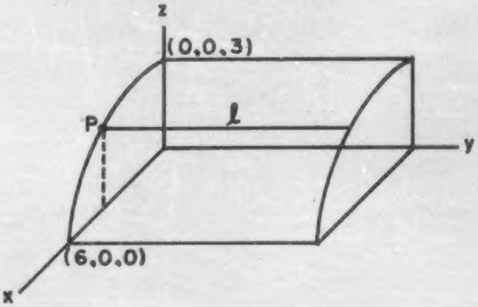


Figure 9-14

Not all cylinders are quadric surfaces. A plane may be considered a cylinder, since one of any two intersecting lines in it may serve as directrix and the other as an element. Other examples of cylinders are the graphs of such equations as $z = \sin y$ and $y = e^x$. You might sketch one of these cylinders.

Exercises 9-4

Discuss and sketch the cylinders represented by equations 1 to 10 .

1. $x^2 + y^2 = 64$
2. $x^2 + z^2 = 25$
3. $y^2 + z^2 = 36$
4. $4x^2 + 9y^2 = 36$
5. $9x^2 + 4z^2 = 36$
6. $4y^2 + 9z^2 = 36$
7. $25x^2 + 144y^2 = 3600$
8. $144x^2 + 25z^2 = 3600$
9. $9x^2 - 4y^2 = 1$
10. $9x^2 - 25y^2 = 1$

11. Write an equation for the locus of points

    (a) at distance 9 from the x-axis

    (b) at distance 6 from the y-axis

    (c) at distance 4 from the z-axis

12. Write an equation for each of the cylinders discribed below.

    (a) Axis is the x-axis, trace in the yz-plane is a circle of radius 3 .

    (b) Axis is the y-axis, trace in the xz-plane is a circle of radius 5 .

    (c) Axis is the z-axis, trace in the xy-plane is a circle of radius 10 .

13. A line moves so that it is always parallel to the y-axis and 10 units from it. What is an equation of its locus?

14. A line moves so that it is always parallel to the x-axis and 12 units from it. What is an equation of its locus?

15. The circle with equations

$$x^2 + z^2 = 4 \ , \ y = 0$$

is the directrix of a cylinder, and a line parallel to the y-axis is an element. What is an equation of the cylinder?

16. Write an equation of the cylinder with the ellipse with equations

$$25y^2 + 4z^2 = 100 \ , \ x = 0$$

as directrix, and a line perpendicular to the yz-plane at a vertex of the ellipse as an element.

Challenge Problems

Discuss and sketch the cylinders represented by Equations 1 to 8 .

1. $x^2 = 4z$	5. $x^2 + z^2 - 6z = 7$
2. $y^2 = z$	6. $x^2 + y^2 + 2x - 4y = 4$
3. $y^2 - z^2 + 1 = 0$	7. $z = \sin x$
4. $xy = 12$	8. $y = \cos z$

9. Write an equation for the cylinder with axis parallel to the x-axis, and with trace in the yz-plane a circle of radius 4 and center at (0,-2,5). Sketch the cylinder.

9-5. The Cone.

Let us investigate the surface whose equation is

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{9} = 0 . \qquad (1)$$

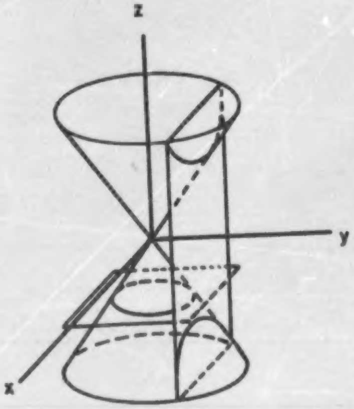


Figure 9-15

When we look for intercepts and the trace in the xy-plane, we find only the point $0 = (0,0,0)$. If $x = 0$, Equation (1) becomes

$$\frac{y^2}{4} - \frac{z^2}{9} = 0 ;$$

the trace in the yz-plane is the union of two intersecting lines. So is the trace in the xz-plane.

We find that the sections parallel to the xy-plane are circles whose radii increase as $|z|$ increases. The sections parallel to the other coordinate planes are hyperbolas. Does this sound familiar? It should, since the surface (Figure 9-15) is a right circular cone, whose sections are the conics we studied in Chapter 7.

A conical surface, or cone, is the surface generated by a line (called an element or generator) which moves so that it always contains a point of a plane curve (called the directrix) and a fixed point (called the vertex) which is not in the plane of the curve. (See Supplement to Chapter 7 for further information on the right circular cone and its sections.) Here we shall consider only right cones with vertex at the origin and the directing curve a conic section in a plane perpendicular to one of the coordinate axes.

As another example, let us sketch the graph of the equation

(2) $$\frac{x^2}{4} - \frac{y^2}{1} + \frac{z^2}{9} = 0 .$$

The sections parallel to the xz-plane are ellipses; the cone (Figure 9-16) is called an elliptic cone.

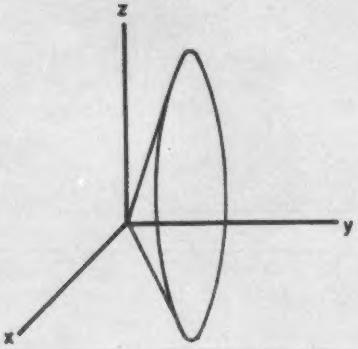


Figure 9-16

Exercises 9-5

Sketch the cones represented by Equations 1 to 6. On each sketch show the intercepts, traces, and at least two of the sections perpendicular to the axis of the cone.

1. $x^2 - z^2 = y^2$
2. $y^2 - z^2 = x^2$
3. $\frac{x^2}{4} - \frac{y^2}{25} + \frac{z^2}{4} = 0$
4. $\frac{x^2}{9} - \frac{y^2}{16} - \frac{z^2}{16} = 0$
5. $4x^2 + 9y^2 - 36z^2 = 0$
6. $16x^2 - 4y^2 + 9z^2 = 0$

Write an equation of each of the cones described in Exercises 7 to 10.

7. Axis is the y-axis, a perpendicular section is a circle whose radius is twice the distance from the origin to the plane of the section.
8. Axis is the x-axis, a perpendicular section at $x = 3$ is an ellipse whose section in that plane is $4y^2 + 9z^2 = 36$.
9. Axis is the z-axis, a perpendicular section at $z = 4$ is a circle of radius 3 .
10. Axis is the y-axis, a perpendicular section at $y = 5$ is an ellipse whose equation in that plane is $9x^2 + z^2 = 16$.

11. It was noted that the sections of the graph of Equation (2) parallel to the xz-plane are ellipses; prove that these ellipses all have the same eccentricity.

Challenge Problems

1. Write an equation of a cone whose axis is the x-axis, and whose sections perpendicular to the axis are ellipses with eccentricity $\frac{2}{3}$. At $x = 1$, the major axis of the ellipse is 12.
2. Write an equation of a cone whose axis is the z-axis, and whose sections perpendicular to the axis are ellipses with eccentricity $\frac{1}{2}$. At $z = 2$, the major axis of the ellipse is 16.

9-6. Surfaces of Revolution.

A surface that is generated by revolving a plane curve about a fixed line in the plane is called a surface of revolution. The fixed line is called the axis of the surface. Some of the quadric surfaces we have discussed here are surfaces of revolution. A sphere is one; it may be generated by revolving any of its great circles about a diameter of that circle. The ellipsoid of Figure 9-8, the paraboloid of Figure 9-9, the cylinder of Figure 9-12, and the cone of Figure 9-15 are all surfaces of revolution.

Let us find the equation of the surface obtained by revolving the parabola with equations $z^2 = 2y$, $x = 0$ about the y-axis. Let $P = (x,y,z)$ be a point on the surface. The plane through P perpendicular to the y-axis intersects the generating curve at the point $C = (0,y,k)$, where $k = d(C,F)$; the same plane intersects the y-axis at the point $F = (0,y,0)$. Since P must lie in this plane on a circle with F as center, its coordinates must satisfy the equation

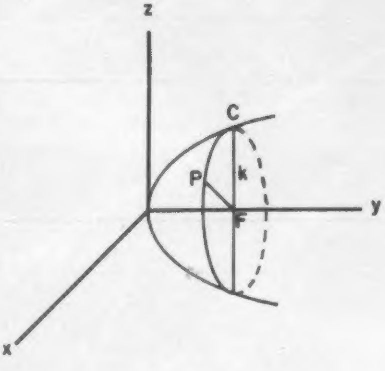


Figure 9-17

$$x^2 + z^2 = k^2 , \tag{1}$$

where k is the radius of the circle. The value of k is determined by the requirement that $C = (0,y,k)$ be on the generating curve $z^2 = 2y$. Therefore,

$$k^2 = 2y\ . \tag{2}$$

Equating the expressions for k^2 in Equations (1) and (2), we have

$$x^2 + z^2 = 2y\ , \tag{3}$$

an equation for the surface of revolution. It is, of course, a paraboloid.

The paraboloid of revolution for which we have just found an equation is generated by a parabola revolving on its axis. The parabola may revolve about lines other than its own axis; suppose it revolves about the z-axis. We sense intuitively that the resulting surface of revolution is quite different. Let us obtain its equation.

We start with equations of the generating curve,

$$z^2 = 2y\ ,\ x = 0\ ,$$

and let $P = (x,y,z)$ be a point on the surface. A plane through P perpendicular to the z-axis intersects the generating curve in $C = (0,k,z)$ where $k = d(C,F)$; the same plane intersects the z-axis in $F = (0,0,z)$.

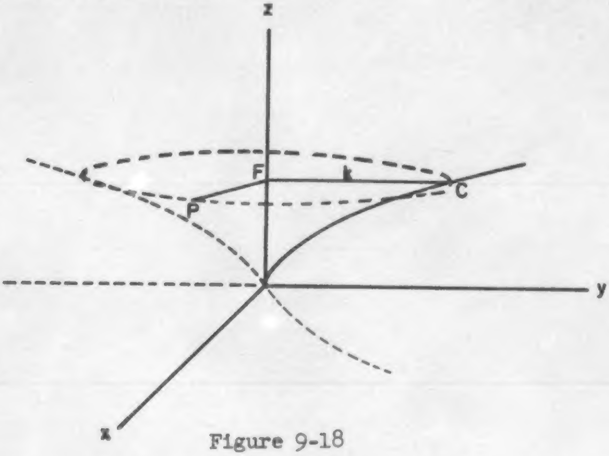


Figure 9-18

Since P lies on a circle in this plane with center F , its coordinates satisfy the equation

$$x^2 + y^2 = k^2\ . \tag{4}$$

Since k is the y-coordinate of C , and C is a point of the generating curve, the coordinates of C must satisfy the equation of that curve; hence

$$z^2 = 2k ,$$

and therefore

$$\frac{z^4}{4} = k^2 . \tag{5}$$

Equating the expressions for k^2 in Equations (3) and (4), we have

$$x^2 + y^2 = \frac{z^4}{4} \tag{6}$$

as an equation of our surface of revolution.

Since Equation (6) is not quadratic, the surface is not a quadric surface. However, we can use the methods of this chapter to investigate its shape. From the equation we see that the surface is symmetric with respect to each of the coordinate planes. Its only intersection with the xy-plane is the origin; the traces in the other coordinate planes are parabolas. The sections parallel to the xy-plane have equations of the form

$$x^2 + y^2 = \frac{k^4}{4} , \; z = k ;$$

clearly they are circles, as they should be for a surface of revolution.

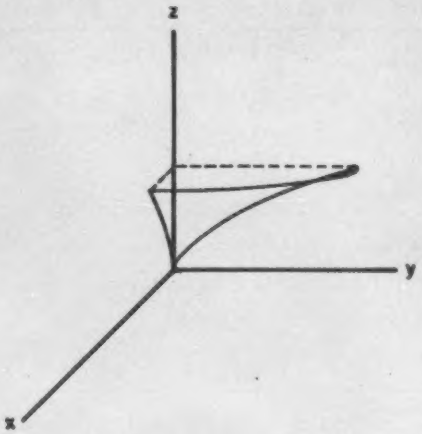


Figure 9-19

Exercises 9-6

In each of Exercises 1 to 18, find an equation of the surface obtained by revolving the plane curve about the axis indicated. Sketch the surface. In Exercises 1 to 10 the curve is to be revolved about its own axis, and the surfaces obtained are quadric surfaces; in Exercises 11 to 18 the axis of revolution is not an axis of the curve.

1. $z^2 = 8y$, $x = 0$; y-axis
2. $x^2 = 2z$, $y = 0$; z-axis
3. $3x = 2y$, $z = 0$; y-axis
4. $3x = 2y$, $z = 0$; x - axis
5. $y^2 + z^2 = 25$, $x = 0$; y-axis
6. $y^2 + z^2 = 25$, $x = 0$; z-axis

7. $9x^2 + 4y^2 = 36$, $z = 0$; x-axis

8. $9x^2 + 4y^2 = 36$, $z = 0$; y-axis

9. $4y^2 - z^2 = 16$, $x = 0$; y-axis

10. $x^2 - 4z^2 = 100$, $y = 0$; x-axis

11. $z^2 = 2x$, $y = 0$; z-axis

12. $x^2 = 2z$, $y = 0$; x-axis

13. $4y^2 - z^2 = 16$, $x = 0$; z-axis

14. $x^2 - 4z^2 = 100$, $y = 0$; z-axis

15. $y^2 = 8z$, $x = 0$; y-axis

16. $36y^2 - 4z^2 = 144$, $x = 0$; z-axis

17. $z = y^3$, $x = 0$; z-axis

18. $z = y^3$, $x = 0$; y-axis

19. If a curve in the yz-plane is represented by the equations $f(y,z) = 0$ and $x = 0$, show that, if $z \geq 0$, an equation of the surface obtained by revolving this curve about the y-axis is

$$f(y, \sqrt{x^2 + z^2}) = 0 .$$

9-7. Intersection of Surfaces. Space Curves.

In order to visualize quadric surfaces we have been discussing the intersections of curved surfaces and planes. This situation is represented by the simultaneous solution of two equations, such as

$$(1) \qquad \begin{aligned} x^2 + y^2 + z^2 &= 25 , \\ z &= 3 . \end{aligned}$$

In this case, by substituting $z = 3$ into the first equation, we have $x^2 + y^2 = 16$, an equation of the circular section of the sphere in the plane $z = 3$. This circle is in a plane parallel to the xy-plane, has its center at $(0,0,3)$, and has radius 4 . It is completely described either by the first pair of equations or, more simply, by the pair

$$(2) \qquad \begin{aligned} x^2 + y^2 &= 16 \\ z &= 3 . \end{aligned}$$

But Equations (2) represent the intersection of a cylinder and a plane. Or we might have

$$(3) \qquad \begin{aligned} \frac{x^2}{16} + \frac{y^2}{16} - \frac{z^2}{9} &= 0 \\ x^2 + y^2 &= 16 , \end{aligned}$$

representing the intersection of a cone and a cylinder. In each case the circle which is the intersection of the two surfaces is the same. You might like to verify this by finding simultaneous solutions. (Equations (3) have an additional solution set.)

It should be intuitively evident by now that there are many pairs of surfaces which intersect in the circle described above. Earlier in your mathematical training you encountered this situation when you described a line as the intersection of two planes. There are infinitely many planes containing a given line, and any two of these planes may be used to describe the line. Similarly, there are infinitely many surfaces passing through a given curve, and this curve may be represented by the equations of any two of the surfaces having this curve as their intersection. Such an intersection is called a space curve. (It is perfectly correct to describe a plane as a surface and a line as a curve.)

From the many representations of a space curve, we try to choose one which gives us immediate information about the shape and location of the curve. For example, Equations (1) tell us at once that the intersection of their graphs is a circle and lies in the plane $z = 3$, but they do not show us the radius or the location of the center of the circle. Equations (3) indicate that the intersection of their graphs is a circle of radius 4, with its center on the z-axis, but we do not immediately see the plane of the circle. All of this information is available at first glance from Equations (2); hence, this representation is likely to be our choice from among the three suggested.

The representation of Equations (2) is useful also in sketching this space curve. Recall that by eliminating the variable z from $x^2 + y^2 + z^2 = 25$ we obtained the equation

$$x^2 + y^2 = 16 , \tag{4}$$

which represents a cylinder whose generators are parallel to the axis of the missing variable, z. Such a cylinder not only contains the curve, but its equation is also the equation of the projection of the curve on the coordinate plane. For this reason, this cylinder is sometimes called a projecting cylinder of the space curve. If the other variables are removed, other projecting cylinders are obtained; since these cylinders contain the curve, any two may be used to show the intersection. Interpreting Equations (2) in this way, we think of the plane $z = 3$ as a cylinder parallel to both the x-axis and the y-axis. For the sketch, we draw the projecting cylinder $x^2 + y^2 = 16$ and show the plane $z = 3$ intersecting it (Figure 9-20).

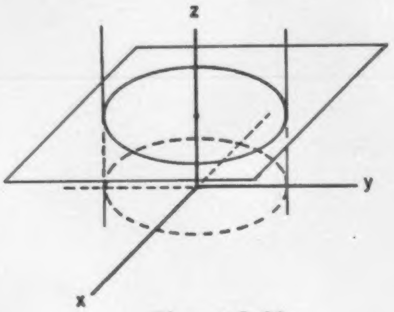


Figure 9-20

Example 1. Find simpler equations for the curve

$$\frac{x^2}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1 ,$$
$$x = 3 .$$

Solution. Let $x = 3$ in the first equation to obtain

$$\frac{9}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1 ,$$

or

$$\frac{y^2}{9} + \frac{z^2}{3} = \frac{2}{3} ,$$

which becomes

$$\frac{y^2}{6} + \frac{z^2}{2} = 1 .$$

The curve is an ellipse represented by

$$\frac{y^2}{6} + \frac{z^2}{2} = 1 ,$$
$$x = 3 .$$

Example 2. A typical problem from calculus could be stated as follows: Find the volume of the region in the first octant bounded by the surfaces $y^2 + z^2 + 2x = 16$, $x + y = 4$, and the coordinate planes.

As a start on this problem, you should make a reasonably accurate sketch of the boundaries of the region. (You can find the volume when you study calculus.) We first find the traces of the surfaces. One surface is a paraboloid of revolution and the other is a plane. Their traces are shown in Figure 9-21. These traces, along with the coordinate axes, provide us with all of the edges of the solid except one. This edge is the space curve which is the intersection of the paraboloid and the plane $x + y = 4$. To find this edge, we eliminate x from the equation of the paraboloid and obtain

$$(y - 1)^2 + z^2 = 9 ,$$

the projecting cylinder parallel to the x-axis. The projection on the yz-plane is a circle with center at $(0,1,0)$ and radius 3, as is shown in the figure. The space curve is represented by

$$(y - 1)^2 + z^2 = 9 \; , \qquad x + y = 4 \; , \tag{5}$$

and we shall now describe how to locate some points on it.

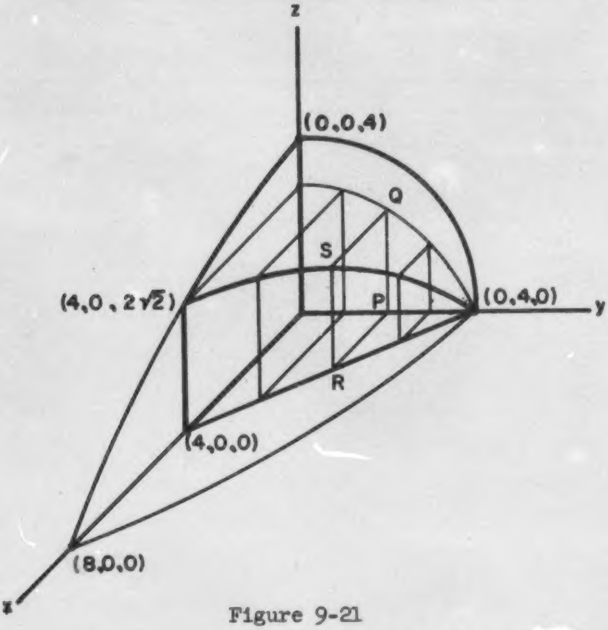


Figure 9-21

Since y is the variable appearing in both equations, we choose a point, P , on the y-axis, and we draw lines parallel to the other axes intersecting the traces of Equations (5) in points Q and R , as shown. We now complete the rectangle by drawing lines parallel to the x- and z-axes from Q and R . These lines intersect at S , a point of the space curve. Other points may be found in a similar manner, and when these points are joined by a smooth curve, the figure is completed.

Example 3. Sketch the curve described by

$$x = 2\cos t \; , \quad y = 2\sin t \; , \quad z = 2t \; .$$

Solution. If we square both members of the first two equations and add, we obtain

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t)$$

or

$$x^2 + y^2 = 4 .$$

This represents a circular projecting cylinder of radius 2 whose axis is the z-axis. All elements of the solution set are contained in this cylinder, and since z is directly proportional to t, we note in Figure 9-22 that the curve is an ascending spiral "wrapping around" the cylindrical surface. This curve is called a helix.

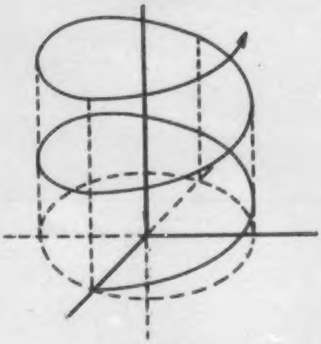


Figure 9-22

We might view this differently by eliminating the parameter t. Then we have

$$x = 2 \cos \frac{z}{2}$$

$$y = 2 \sin \frac{z}{2} ,$$

and the curve is seen to be the intersection of two projecting cylinders whose cross-sections are sine (or cosine) curves. The elements of one cylinder are parallel to the y-axis; the elements of the other cylinder are parallel to the x-axis. If you wish to build a model for this problem, you might use two pieces of corrugated cardboard.

Still another view of this curve may be obtained by writing the equations in cylindrical coordinates. We shall consider this in the next section.

Exercises 9-7

1. Name and describe the intersection of each of the following pairs of equations, and write for each a simpler pair (if there is one).

   (a) $x^2 + y^2 + z^2 = 16$,
   $y = -2$.

   (b) $x^2 + y^2 + z^2 = 4$,
   $x = 3$.

(c) $x^2 + y^2 = 4$,
$z = 0$.

(d) $x^2 + y^2 + z^2 = 4$,
$z = 0$.

(e) $x^2 + z^2 = 25$,
$y = 5$.

(f) $x^2 + z^2 = 25$,
$z = 0$.

(g) $x^2 + y^2 = 50$,
$x - y = 0$.

(h) $x^2 + 8y^2 - 4z^2 = 12$,
$z = 1$.

(i) $x^2 + 3y^2 - 4z^2 = 12$,
$x = 0$.

(j) $x^2 + 2y^2 + 8z^2 = 8$,
$x = 0$.

(k) $x^2 + 2y^2 + 8z^2 = 8$,
$y = 2$.

(ℓ) $x^2 + y^2 - z^2 = z$,
$x^2 + y^2 - z^2 = 1$.

2. Make a sketch of the region in the first octant bounded by the given surfaces and the coordinate planes.

(a) Inside the cylinder $x^2 + y^2 = 50$ and under the plane $x + y + z = 10$.

(b) Inside the cylinder $y^2 + z^2 = 16$ and in the half-space formed by $x + y = 6$ which contains the origin.

(c) Inside the paraboloid $x^2 + y^2 = 4z$ and under the plane $z = 2$.

(d) Inside the cylinder $y^2 + z^2 = 25$ and inside the cylinder $x^2 + z^2 = 25$.

(e) Inside the sphere $x^2 + y^2 + z^2 = 25$ and inside the cylinder $y^2 + z^2 = 16$.

(f) Under the paraboloid $18z = 4x^2 + 9y^2$ and in the half-spaces formed by $x = 2$ and $y = 3$ which contain the origin.

3. Find the equations of the projecting cylinders of the curve whose equations are

$$x^2 + 2y^2 - z^2 = 3 ,$$
$$x^2 + y^2 - 2z^2 = -3 .$$

Sketch the curve by making use of the projecting cylinders.

* 4. A calculus problem requires the student to find the height above the xy-plane in which the plane $2x + y = 2$ intersects the paraboloid $z = 16 - 4x^2 - y^2$. Find this height by sketching in one of the coordinate planes the trace of a projecting cylinder.

* 5. A calculus problem asks for the volume inside the cylinder $x^2 + y^2 - 2y = 0$ and between the xy-plane and the upper nappe of the cone $z^2 = x^2 + y^2$. Make a sketch for this problem, showing the portion of the region in the first octant.

9-8. Cylindrical and Spherical Coordinate Systems.

Some problems in science that have a setting in 3-space are easier to handle if they are expressed in terms of cylindrical or spherical coordinates. If the surface has symmetry with respect to a line, then cylindrical coordinates may simplify the work of the problem. If the surface has point-symmetry, the use of spherical coordinates may provide a simpler analytic representation and solution.

Cylindrical Coordinates are a combination of polar and rectangular coordinates. A polar coordinate system is used in one coordinate plane; the axis perpendicular to this plane has a linear coordinate system. A point is designated in cylindrical coordinates by an ordered triple. We use (r,θ,z), as indicated in Figure 9-23. The first two coordinates are the coordinates of the projection of P in the polar plane. The third coordinate is the coordinate of the projection of P on the linear axis. In this figure we may verify what

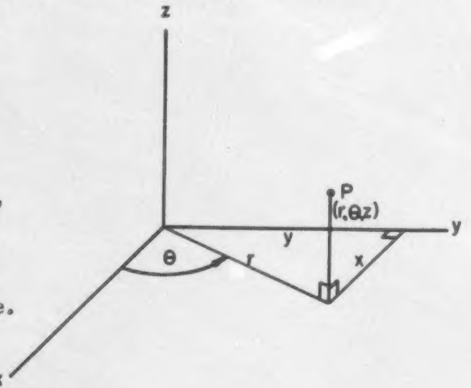


Figure 9-23

we could have guessed; the transformations from cylindrical to rectangular form, and vice versa, are accomplished by the same process we used in Section 2-4 to relate polar and rectangular coordinates. The transforming equations are

$$\begin{aligned} x &= r\cos\theta & \qquad r^2 &= x^2 + y^2 \\ y &= r\sin\theta & \qquad \tan\theta &= \frac{y}{x}\,,\ \text{where}\ \ x \neq 0 \\ z &= z & \qquad z &= z \end{aligned}$$

The simple equation, $r = k$, represents, in cylindrical coordinates, a right circular cylinder with radius k whose axis is the linear axis. This fact accounts for the name applied to this system.

Example 1. Write in cylindrical coordinates the equation of the sphere with radius $\sqrt{5}$ whose center is at the origin.

Solution. In rectangular coordinates the equation is $x^2 + y^2 + z^2 = 5$. Since $r^2 = x^2 + y^2$, the equation is written $r^2 + z^2 = 5$.

Example 2. Transform to rectangular coordinates and identify the surface whose equation in cylindrical coordinates is $3r\cos\theta + r\sin\theta + 2z = 0$.

Solution. Using the transforming equations, we obtain $3x + y + 2z = 0$, the equation of a plane.

Example 3. In connection with the helix in Example 3 of the previous section, we suggested a solution using cylindrical coordinates. We write θ in place of t, use the transforming equations, and square as before, obtaining

$$r^2 = x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta\,,$$

$$r^2 = 4(\cos^2\theta + \sin^2\theta)\,,$$

or

$$r^2 = 4\,.$$

Since $r = 2$ has the same graph as $r^2 = 4$, we obtain a simple expression for the helix:

$$r = 2$$
$$r = 2\theta \ .$$

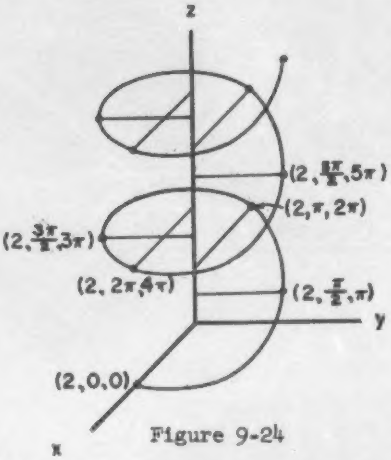

z
$(2, \frac{5\pi}{2}, 5\pi)$
$(2, \pi, 2\pi)$
$(2, \frac{3\pi}{2}, 3\pi)$
$(2, 2\pi, 4\pi)$
$(2, \frac{\pi}{2}, \pi)$
y
(2,0,0)
x

Figure 9-24

Since this helix is a constantly ascending spiral around the z-axis, we can locate some of its points by a device we might describe as fixing "ribs" to a "spine", or of locating steps on a spiral staircase. The z-axis will be the "spine" to which the "ribs" are attached. (We are using a condensed scale on the z-axis to save space.)

We first locate a point at $(2,0,0)$ as shown in Figure 9-24. When $\theta = \frac{\pi}{2}$, we have rotated to a point one-quarter of the way around the "spine", and we have ascended a distance π. We fix a "rib" to this point. We might next stop at $\theta = \pi$ and fix another point. This process can be continued as long as desired and the points may be connected by a smooth curve to sketch a portion of the helix.

Another useful system for locating points in 3-space involves the use of spherical coordinates. In this system the coordinates of a point P are determined by assuming a polar coordinate system in the plane determined by the point P and the z-axis. The positive half of the z-axis is the polar axis and the positive sense of the polar angle is from the polar axis to ray $\overrightarrow{OP}$. The polar distances $d(O,P)$ is denoted by ρ and the measure of the polar angle by ϕ. In the xy-plane the usual system of polar angles is assumed. The projection of P in the xy-plane determines the terminal side of a polar angle of measure θ. These three numbers represent the point P and are called the spherical coordinates of P. They are written as an ordered triple, usually as (ρ,θ,ϕ). In Figure 9-25 this system is used to name the point which in rectangular coordinates would be $P = (x,y,z)$.

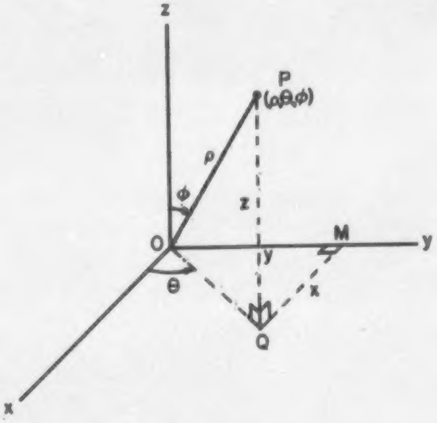


Figure 9-25

In order to relate spherical coordinates and rectangular coordinates, we obtain (from Figure 9-25) the following relations:

$$x = d(Q,M) = d(O,Q) \cos \theta = \rho \sin \phi \cos \theta ,$$
$$y = d(O,M) = d(O,Q) \sin \theta = \rho \sin \phi \sin \theta ,$$
$$z = \rho \cos \phi .$$

The derivation of the equations for relating spherical coordinates and cylindrical coordinates is left as an exercise.

Example 1. Write in spherical coordinates the equation of the sphere with radius $\sqrt{5}$ whose center is at the origin.

Solution. Since ρ is the distance from the origin to a point, we obtain

$$\rho = \sqrt{5} .$$

This simple equation form, $\rho = k$, for a sphere in spherical coordinates accounts for the name applied to this system. Compare this with $r = k$ in cylindrical coordinates and $r = k$ in polar coordinates.

Example 2. Transform to rectangular or cylindrical coordinates and identify the surface whose equation in spherical coordinates is $\rho \sin \phi = 3$.

Solution. We square both members and obtain

$$\rho^2 \sin^2 \phi = 9 .$$

Multiplying the left member by 1 (disguised as $\cos^2 \theta + \sin^2 \theta$), we have

$$\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) = 9 ,$$
$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 9 ,$$

which in rectangular coordinates is

$$x^2 + y^2 = 9 .$$

In cylindrical coordinates we have simply

$$r = 3 .$$

This is the equation of a right circular cylinder with radius 3 whose axis is the z-axis.

It may come as a surprise when you realize that very likely you used spherical coordinates before you knew what they were. In terms of the position of a point on the earth, θ is the longitude, $90^\circ - \phi$ is the latitude, and (assuming the earth is a sphere) ρ is the earth's radius.

Exercises 9-8

1. Derive transforming equations to relate cylindrical coordinates and spherical coordinates.

2. Write the rectangular and the cylindrical coordinates of the points whose spherical coordinates are

   (a) $(4, \frac{\pi}{3}, \frac{\pi}{4})$.

   (b) $(3, 0, \frac{\pi}{3})$.

   (c) $(2, \frac{\pi}{2}, \frac{\pi}{2})$.

   (d) $(4, \frac{3}{2}, 1)$.

3. Write the rectangular and the spherical coordinates of the points whose cylindrical coordinates are

   (a) $(2, \frac{\pi}{6}, 3)$.

   (b) $(5, \frac{\pi}{2}, 0)$.

   (c) $(0, \frac{\pi}{4}, 8)$.

   (d) $(4,1,2)$.

4. Write the cylindrical and the spherical coordinates of the points whose rectangular coordinates are

   (a) $(2,3,0)$.
   (b) $(0,6,3)$.
   (c) $(2\sqrt{3},2,4)$.
   (d) $(4,1,2)$.

5. Transform the following equations into cylindrical coordinates and into spherical coordinates.

   (a) $x^2 + y^2 = 25$.

   (b) $xz = 4y$.

   (c) $x^2 + y^2 = 8x$.

   (d) $x^2 + y^2 = 3z$.

6. Transform the following equations into rectangular coordinates.

   (a) $\rho = 6$.

   (b) $r = 6$.

   (c) $z = 6 + r$.

   (d) $z^2 = 9 - r^2$.

7. Identify and describe each of the following surfaces.

   (a) $r = 3$.

   (b) $\theta = \frac{\pi}{4}$.

   (c) $r^2 + z^2 = 4$.

   (d) $\phi = \frac{\pi}{4}$.

   (e) $\rho \cos \phi = 7$.

   (f) $z = r \cos \theta$.

   (g) $z = r$.

   (h) $r = 2 \sec \theta$.

8. A circular cylinder of diameter 4 intersects a sphere of radius 4 so that an element of the cylinder contains a diameter of the sphere. Choose axes and write equations of the bounding surfaces in

   (a) rectangular coordinates,

   (b) cylindrical coordinates, and

   (c) spherical coordinates.

9-9. Summary.

Our work in this chapter has been limited to the most important and familiar quadric surfaces, and we have located the coordinate axes so as to get simple equations for them. Students who have enjoyed this work may like to pursue it further by looking up such topics as ruled surfaces, hyperbolic paraboloids, curves in space, and surfaces of higher order.

Our objective here has been to develop methods to help you visualize surfaces and curves in space. The methods we have used are general, and should be of use to you in visualizing or sketching, particularly in your work in calculus and its applications.

Surfaces in space are represented by one equation, $f(x,y,z) = 0$; for quadric surfaces, the equation is of the second degree. Curves in space are given by the intersection of two equations (or three in parametric form), $f(x,y,z) = 0$ and $g(x,y,z) = 0$. The most important curves for sketching a surface are the traces and the sections parallel to the coordinate planes.

The surfaces we have studied include the cone, cylinder, sphere and ellipsoid, elliptic paraboloid, and the hyperboloid. A cone is generated by a line moving about a line with one point fixed, a cylinder by a line moving parallel to a fixed line, and a surface of revolution by a plane curve revolving about a line in the plane of the curve. For the limited cases we have studied, the quadric surfaces may be identified by their sections parallel to the coordinate planes as follows:

Quadric Surface	Sections Parallel to Coordinate Planes
Cone	Conic sections, including degenerate cases.
Elliptic or circular cylinder	Central ellipses or circles, parallel lines, or a line.
Sphere	Circles, including point-circle.
Ellipsoid	Ellipses, including circles and points.
Elliptic paraboloid	Parabolas and ellipses, including circles and points.
Hyperboloid	Ellipses, including circles and points, and hyperbolas

In sketching a surface, $f(x,y,z) = 0$, it is suggested that information about it be obtained and placed on the graph in the following order:

1. Intercepts — Set two of the variables equal to zero and solve the resulting equation for the third variable to find the intercepts on each axis.
2. Traces — Let the variables equal zero, one at a time, to find the equations of the traces - the sections in the coordinate planes.

3. Sections Let $z = k$, where k is a constant, to find the sections parallel to the xy-plane, for example. You can build up a sketch of the figure by using enough different values of k . For this purpose, select the sections easiest to draw.

We determine symmetry with respect to the xy-, yz-, or xy-plane by checking that the equation of the surface is unchanged when $-z$, $-x$, or $-y$ is substituted for z , x , or y , respectively. Knowing the symmetries of a surface helps in identifying it and sketching it. When a surface is symmetric, we often draw only the part in the first octant.

Certain curves which are the intersection of two surfaces were studied. In addition to using intercepts and traces, we used projecting cylinders to help us visualize and draw space curves.

Finally, cylindrical and spherical coordinates were introduced as other ways of describing the location of points in space.

Review Exercises

Discuss and sketch the surfaces represented by the equations in 1 to 20.

1. $16x^2 + 9y^2 + 16z^2 = 144$
2. $5x^2 + 5y^2 + 5z^2 - 45 = 0$
3. $16z = x^2 + y^2$
4. $36z = 9x^2 + 4y^2$
5. $25x^2 + 100y^2 = 400z$
6. $16x^2 + 9y^2 + 9z^2 = 144$
7. $9x^2 + 9y^2 + 9z^2 - 16 = 0$
8. $4x^2 - 9y^2 + 4z^2 = 36$
9. $4x^2 + 9z^2 = 36$
10. $4x^2 - 9z^2 = 36$
11. $9x^2 - 4y^2 = 0$
12. $36y^2 + 25z^2 = 900x$
13. $-16x^2 + 25y^2 + 16z^2 = 400$
14. $y^2 + z^2 = 100$
15. $x^2 + y^2 + z^2 - 2x - 3 = 0$
16. $25x^2 + 25y^2 + 25z^2 = 0$
17. $16x^2 - 9y^2 + 9z^2 = 0$
18. $x^2 + y^2 + z^2 + 8x - 6y + 10z + 34 = 0$
19. $36x^2 + 25z^2 = 900$
20. $25x^2 - 9y^2 - 9z^2 = 0$.

Discuss and sketch the surfaces described in Exercises 21 to 38. Write an equation for each surface; identify those that are not named.

21. A sphere centered at the origin with radius 10.

22. An ellipsoid with axes of lengths 12, 10, and 8.

23. A circular cylinder with radius 5 and axis the x-axis.

24. A prolate spheroid with axes of lengths 4 and 16.

25. An oblate spheroid with axes of lengths 4 and 6.

26. A cylinder with the y-axis as its axis, and its trace in the xz-plane the ellipse with equation $25x^2 + 16z^2 = 400$.

27. The surface obtained by revolving the curve with equations $16x^2 - 9y^2 = 144$, $z = 0$ about the y-axis.

28. The surface obtained by revolving the curve with equations $x^2 = 4z$, $y = 0$ about the z-axis.

29. The surface obtained by revolving the curve with equations $z^2 = 8y$, $x = 0$ about the y-axis.

30. The surface obtained by revolving the curve with equations $25x^2 - 36z^2 = 900$, $y = 0$ about the x-axis.

31. Refer to Exercise 27, but revolve about the x-axis.

32. Refer to Exercise 28, but revolve about the x-axis.

33. Refer to Exercise 29, but revolve about the z-axis.

34. Refer to Exercise 30, but revolve about the z-axis.

35. The surface obtained by revolving the curve with equations $25x^2 - 16y^2 = 0$, $z = 0$ about the x-axis.

36. Refer to Exercise 35, but revolve about the y-axis.

37. The surface obtained by revolving the line with equations $x = 2$, $y = 0$ about the z-axis.

38. Refer to Exercise 37, but revolve the line with equations $x = 2z$, $y = 0$.

39. Write an equation for the locus of poihts 10 units from $P = (3,-2,1)$.

40. Write an equation for the locus of points 5 units from the y-axis.

41. Write an equation for the locus of points equidistant from the plane $x = 0$ and the point $(6,0,0)$.

42. What are the graphs of the following equations?

(a) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

(b) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 0$

(c) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = -1$

(d) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

(e) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0$

(f) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = -1$

(g) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1$

(h) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 0$

(i) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = -1$

43. Points A and B are 4 units apart. Write an equation for the locus of a point the sum of whose distances from A and B is 6. Simplify the equation, sketch the graph, and identify it.

44. Follow the same instructions as in the previous exercise, but let the difference of the distances be 2.

45. A pencil with a hexagonal cross-section is sharpened. Describe the space curve which you see as the edge of the painted surface of the pencil.

46. A cube having edges 1 unit in length has one vertex at the origin and three of its faces each in one of the coordinate planes. A plane contains the midpoint of the diagonal of the cube from the origin and is perpendicular to the diagonal. Find the sections of this plane on the faces of the cube. What kind of figure is this set of sections?

47. Sketch the intersection of the surfaces

$$x^2 + y^2 + z^2 = 4 , \quad x^2 + y^2 - 4y = 0$$

in the first octant, using projecting cylinders.

48. In each of the following cases, classify the given surfaces, find the projecting cylinders of the curve of intersection, and sketch the curve.

(a) $x^2 + 2y^2 + z^2 = 8 , \; 3x^2 + 2y^2 - z^2 = 8$.

(b) $x^2 + 2y^2 + z^2 = 4 , \; -2x^2 - y^2 + z^2 = 2$.

(c) $x^2 + y^2 + z^2 = 1 , \; x^2 + y^2 + 2z^2 = 5$.

(d) $x^2 + y^2 = z , \; x^2 + y^2 = 4$.

49. Sketch the solid in the first octant bounded by the given surfaces and the coordinate planes.

(a) $x^2 + z^2 = 1$, $y = 2$.

(b) $y = x$, $z = x + y$, $x = 1$.

(c) $x^2 + y^2 = 9$, $z = y$, $z = 2y$.

(d) $x^2 + y^2 = 36$, $x^2 + z^2 = 25$.

50. Express each equation in terms of two other coordinate systems. (Assume that all relate to 3-space.)

(a) $z = 5$.

(b) $x^2 + y^2 = 4x$.

(c) $r = 7$.

(d) $x^2 + y^2 + z^2 = 25$.

(e) $r^2 + z^2 = 9$.

(f) $\rho \cos \phi = 6$.

(g) $x^2 - y^2 = 16$.

(h) $r = 2 \cos \theta$.

(i) $\rho \sin^2 \phi = 2 \cos \phi$.

(j) $\rho \sin \phi = 3$.

(k) $x^2 + y^2 = 64$.

(ℓ) $\rho \sin \theta \cos \phi = \cos \theta$.

Challenge Problems

Describe and sketch the surfaces represented by Equations 1 to 6 .

1. $z = \sin y$

2. $y = \cos x$

3. $z = x^2 - 2x$

4. $4x^2 + 9y^2 + 36z^2 + 8x - 54y - 72z = 23$

5. $x^2 + y^2 - 4z^2 + 2x + 6y + 8z = 10$

6. $z = \dfrac{x^2 - y^2}{x^2 + y^2}$

Chapter 10

GEOMETRIC TRANSFORMATIONS

10-1. Why Study Geometric Transformations?

In previous chapters you have had considerable experience in relating a graph and its analytic representation. Because of their importance, conic sections were given very careful treatment. Despite this emphasis you may have noticed that, with the exception of the circle, all the conics you sketched had their centers, foci, vertices at the origin and one or both of the coordinate axes as axes of symmetry.

However, in various studies where the graphs of the equations of conics (and other curves) are of importance, one encounters more complicated analytic representations of these curves. Consider, for example, the following pairs of equations:

(1) $x^2 + y^2 + 10x - 4y + 4 = 0$, $x^2 + y^2 = 25$;

(2) $x^2 - y^2 - 4x - 6y - 30 = 0$, $x^2 - y^2 = 25$;

(3) $y^2 - x - 6y + 11 = 0$, $y^2 = x$.

If you went to the trouble of graphing all six of these equations, you would find that each pair of equations represents a pair of congruent graphs. They differ only in their placement with respect to their coordinate axes. If one is interested in geometric properties of such graphs, it is clear that the second equation of each pair is simpler to analyze and will quite readily yield information regarding intercepts, symmetry, asymptotes, etc., relative to its coordinate system.

It is one of the purposes of this chapter to show how we can relate such a complicated equation of a curve to a simpler equation of the same curve represented in a different coordinate system. The operation which performs this task (among others) is commonly referred to as either a "transformation of axes" or a "transformation of coordinates".

In this chapter we will consider two types of transformations which accomplish the purpose just described. The type we treat first (in Sections 2 and 3) is one wherein the operation is performed on the axes and the graph under study remains fixed. We then turn our attention (in Sections 5 and 6) to the type wherein the operation is performed on the points of the curve while the axes remain fixed. We refer to the latter type as a point transformation.

Our task takes on one of two aspects. We may be given a relationship between the coordinates of $P = (x,y)$ on a curve C and the coordinates of $P' = (x',y')$ on a curve C' and then investigate the correspondence between C and C'. On the other hand, the converse is considered: Given two curves C and C' and some correspondence between them, we investigate the manner in which the coordinates of any point $P = (x,y)$ on C are related to the coordinates of the corresponding point $P' = (x',y')$ on C'.

In the cases of the three pairs of equations presented earlier, the corresponding curves were actually congruent and the point correspondence was one-to-one. In other cases the corresponding curves need not be congruent although there may still be significant relations between them. For example, in Section 6, you will encounter a correspondence between a straight line and a circle under a transformation called an inversion.

Certain transformations preserve geometric properties such as the measure of distance between points on the original curve, the measure of angle between two lines, the order of points on a line, etc. ... while others do not preserve these properties. Discovering which geometric properties are invariant (do not change) under a set of transformations is of significance to the advanced students of geometry because these properties help them to classify the large number of geometries which have been created. This topic is discussed in Section 4.

We may also speak of the properties of a transformation. An important transformation we shall meet in Section 6 has the property that it preserves measure of angle but not necessarily measure of distance. Transformations which have this property are called conformal and have many applications in science.

10-2. Translations.

Suppose we have a curve in the coordinate plane and an equation of the curve. Let us consider the problem of writing an equation of the same curve with respect to another pair of axes. The process of changing from one pair of axes to another is called "*transformation of axes*" or "a *transformation of coordinates*" as stated earlier.

One of the most useful, as well as simple, transformations is one in which the new axes are shifted in such a way that they remain parallel to their original positions and oriented in the same direction. Such a transformation is called a *translation*.

THEOREM 10-1. Given a coordinate system in a plane with origin at O. The axes are then translated so that the new origin is at $O' = (h,k)$. If (x,y) and (x',y') are the coordinates of a point P when referred to the original and new axes respectively, then $x' = x - h$ and $y' = y - k$.

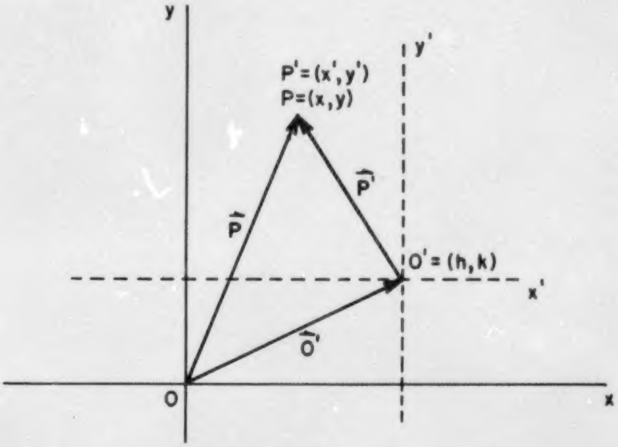


Figure 10-1

Proof. Let $\overrightarrow{P} = [x,y]$, $\overrightarrow{O'} = [h,k]$ and $\overrightarrow{P'} = [x',y']$.

(1) $\overrightarrow{P} = \overrightarrow{O'} + \overrightarrow{P'}$.

(2) $$\begin{aligned}[x,y] &= [h,k] + [x',y'] ,\\ &= [h + x', k + y'] .\end{aligned}$$

(3) Thus $$\begin{cases} x = x' + h \\ y = y' + k . \end{cases}$$ (Why?)

If we solve these equations for x' and y' , we obtain the "inverse form":

(4) $$\begin{cases} x' = x - h \\ y' = y - k . \end{cases}$$

We shall refer to the Equations (3) or (4) as the equations of translation.

Example 1. Find the new coordinates of the points $P_1 = (-3,1)$, $P_2 = (4,-2)$ if the origin is moved to $(-3,5)$.

Solution: Since $h = -3$, $k = 5$, the equations of translation are:

$$\begin{cases} x' = x + 3 \\ y' = y - 5 . \end{cases}$$

Applying these equations, we see that the point $P_1 = (-3,1)$ now has the coordinates $(0,-4)$, and $P_2 = (4,-2)$ now has the coordinates $(7,-7)$ with respect to the new axes.

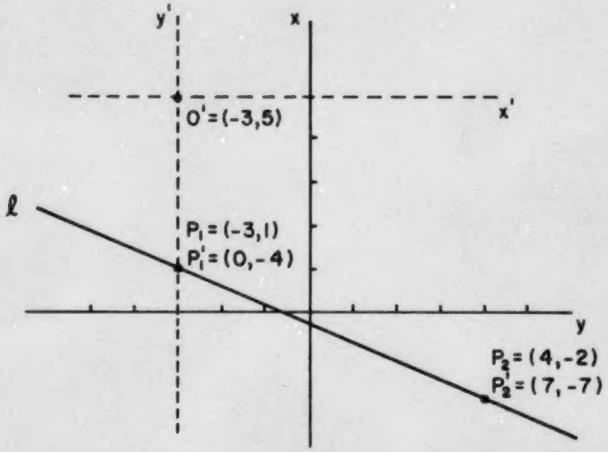


Figure 10-2

Consider an equation of a curve $f(x,y) = 0$. By the equations of translation, the coordinates x and y are transformed respectively into $x' + h$ and $y' + k$. Thus the equation $f(x,y) = 0$ changes to $f(x' + h, y' + k) = 0$. The two equations represent the same curve since the point $P(x,y)$ whose coordinates satisfy $f(x,y) = 0$ is the same as the point $P' = (x',y')$ whose coordinates satisfy $f(x' + h, y' + k) = 0$.

To illustrate this, consider the line ℓ in Figure 10-2 passing through the points P_1 and P_2 of Example 1. The equation of line ℓ is $3x + 7y + 2 = 0$. We now replace x by $x' - 3$ and y by $y' + 5$ and the equation of ℓ is now $3x' + 7y' + 28 = 0$. We note that the coordinates of points $P_1' = (0,-4)$ and $P_2' = (7,-7)$ satisfy this last equation. The new equation $3x' + 7y' + 28 = 0$ represents the same line, with respect to the new axes, x' and y', with the new origin at $O' = (-3,5)$.

Example 2. Find the equation of the circle $x^2 + y^2 + 10x - 4y + 4 = 0$ after a translation moves the origin to the point $(-5,2)$.

Solution: The equations of translation are $x = x' - 5$, $y = y' + 2$. Substituting into the equation of the circle, we have

$$(x' - 5)^2 + (y' + 2)^2 + 10(x' - 5) - 4(y' + 2) + 4 = 0 .$$

If we expand and collect terms, our equation simplifies to $x'^2 + y'^2 = 25$. We infer immediately that the circle has a radius of 5 units and that its center is at $O' = (-5,2)$. If you were to find the locus (or graph) of the original equation, you would discover that you had precisely the same circle. After doing this, you would appreciate the advisability of translating the axes. Note that the principal difference in the two equations is that one contains first degree terms and the other does not.

The basic question is: How do we know where to place the new origin so that a complicated equation reduces to a simple one? This method is illustrated in Example 3.

Example 3. Translate the axes so that the equation of the circle $x^2 + y^2 + 10x - 4y + 4 = 0$ can be written in a form which contains no first degree term.

Solution:

(1) Write the equation in the form $x^2 + 10x + y^2 - 4y = -4$ and complete the squares as follows:

$(x^2 + 10x + 25) + (y^2 - 4y + 4) = 4 + 25 - 4$ or

$(x + 5)^2 + (y - 2)^2 = 25$.

(2) If we let $x' = x + 5$ and $y' = y - 2$, our last equation becomes $x'^2 + y'^2 = 25$.

(3) We note that the equations $\begin{cases} x' = x + 5 \\ y' = y - 2 \end{cases}$ are the equations of translation to new axes with the origin at $(-5,2)$.

To show the wider applicability of this method, let us do one more example:

Example 4. Graph the curve $4x^2 - 9y^2 + 40x + 36y + 28 = 0$.

Solution.

(1) Rewrite the equation in the following form so that we can use the method of "completing the square":

$$4(x^2 + 10x) - 9(y^2 - 4y) = -28 .$$

(2) Completing the square:

$$4(x^2 + 10x + 25) - 9(y^2 - 4y + 4) = -28 + 100 - 36$$

or $$4(x + 5)^2 - 9(y - 2)^2 = 36 .$$

(3) Substituting $x' = x + 5$ and $y' = y - 2$, we have

$$4x'^2 - 9y'^2 = 36$$

or $$\frac{x'^2}{9} - \frac{y'^2}{4} = 1 .$$

We recognize this curve to be a hyperbola with center $O' = (-5,2)$. This curve can now be drawn by using the methods discussed in the earlier chapters.

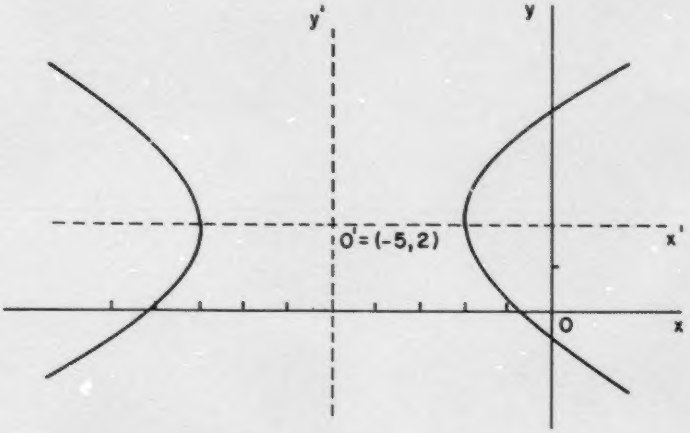


Figure 10-3

The translation of axes can be used to simplify equations of curves other than conics, but at this point we will restrict our discussions to such curves.

We will now generalize the above:

(1) A circle in the form $(x - h)^2 + (y - k)^2 = r^2$ can be simplified to $x'^2 + y'^2 = r^2$.

(2) An ellipse in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ can be simplified to $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$.

(3) A hyperbola in the form $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ can be simplified to $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$.

(4) A parabola in the form $(y - k)^2 = 4p(x - h)$ or $(x - h)^2 = 4p(y - k)$ can be simplified to $y'^2 = 4px'$ or $x'^2 = 4py'$ respectively.

(5) The equilateral hyperbola $(x - h)(y - k) = c$ can be simplified to $x'y' = c$.

All of the above can be done by translating the axes to a new origin at $O' = (h,k)$ by use of the equations of translation

$$\begin{cases} x = x' + h \\ y = y' + k \, . \end{cases}$$

Exercises 10-2

1. Write the equations of translation which change the coordinates of $A = (2,12)$ to $(5,8)$ with respect to a new origin O' . What are the coordinates of O' with respect to the first origin?

2. Determine the equation of the curve represented by $2x^2 - y^2 - 12x - 4y + 12 = 0$ if the origin is translated to $(3,-2)$.

3. Given the transformation $\begin{cases} x' = x + 4 \\ y' = y + 6 \end{cases}$.

   What effect does this transformation have when it is applied to the curves:

   (a) $x^2 + y^2 = r^2$?

   (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$?

4. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are vertices of a right triangle. Find the coordinates of these points if the origin is moved to $O' = (-4,-2)$ by a translation of axes. Using the new coordinates give two proofs that an observer at O' can present to demonstrate that $\triangle ABC$ is a right triangle.

5. Translate the axes so that the equation of the curve $x^2 - y^2 + 10x + 4y + 5 = 0$ can be written in a form containing no first degree terms. Indicate the equations of translation, draw both sets of axes, and sketch the curve.

6. Given circle $Q : x^2 + y^2 = 25$. Find the coordinates of three points A , B , and C on this circle. Then find their coordinates if the origin is translated to $O' = (1,-1)$ and the equation of the circle with respect to O' . Verify that the new coordinates of A , B , and C satisfy the transformed equation.

7. A line L has the equation $3x - 2y + 6 = 0$. Draw the line. The axes are then translated twice in succession in accordance with the equations

$$(1) \quad \begin{cases} x = x' + 3 \\ y = y' + 2 \end{cases} \quad \text{followed by} \quad (2) \quad \begin{cases} x' = x'' + 4 \\ y' = y'' + 5 . \end{cases}$$

Find the equation of L with respect to both the x'- and y'- and x''- and y''- axes. Then find the equations of translation which would perform both operations at once. What would be the effect of commuting translations (1) and (2) ?

8. Sketch the curves after performing a convenient translation of axes. Indicate the equations of translation and draw both sets of axes.

   (a) $y^2 - 6y - 12x - 3 = 0$

   (b) $3x^2 + 4y^2 - 6x + 8y - 5 = 0$

   (c) $2x^2 + 6x - 3y + 12 = 0$

   (d) $(x + 3)(y - 4) - 12 = 0$

   (e) $(y + 2)^2 = (x + 2)^3$

9. Derive the equations for the translation of axes with the new origin at $O' = (h,k)$ without the use of vectors.

10-3. Rotation of Axes: Rectangular Coordinates.

We next consider a rotation of a rectangular coordinate system C . We introduce a new coordinate system C' whose origin coincides with the origin of C and whose axes are obtained by rotating the axes of C through an angle α . Thus α is an angle in standard position whose initial side is the positive side of the x-axis and whose terminal side is the positive side of the x'-axis. Once again we want to discover the relationship between the coordinates of a point P in C and the coordinates of the same point in C'.

The presence of the angle α suggests the use of polar coordinates. We consider the systems of polar coordinates associated with C and C' by letting the polar axes be the positive sides of the x-axis and the x'-axis. Thus, as we have seen in Chapter 2 , if P is the point (r,θ) in the polar coordinate system whose polar axis is the positive side of the x-axis, then

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{1}$$

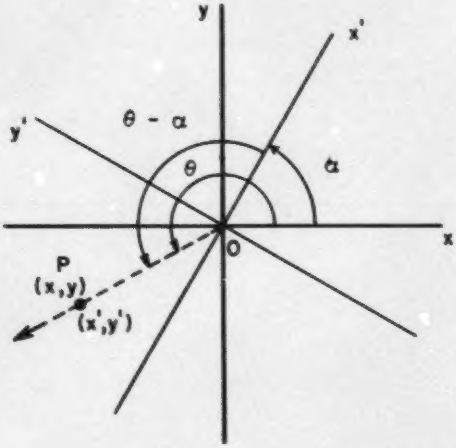


Figure 10-4

However, in the polar coordinate system whose polar axis is the positive side of the x'-axis, P is clearly the point $(r, \theta - \alpha)$. Therefore,

$$\begin{aligned} x' &= r\cos(\theta - \alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ y' &= r\sin(\theta - \alpha) = r(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \end{aligned} \tag{2}$$

Combining equations (1) and (2), we get

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha \end{aligned} \tag{3}$$

These transformation equations are often called equations of rotation.

Example 1. In a given coordinate system, two points P_1 and P_2 have the coordinates (2,3) and (-4,5) respectively. The axes are then rotated through an angle of 30°. Find the rectangular coordinates of P_1 and P_2 with respect to the new axes.

Solution: Since $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$, we have upon substitution in the preceding equations,

$$\begin{cases} x' = \frac{1}{2}(\sqrt{3}x + y) \\ y' = \frac{1}{2}(-x + \sqrt{3}y) \, . \end{cases}$$

Thus P_1 has the new coordinates $\left(\frac{2\sqrt{3} + 3}{2}, \frac{-2 + 3\sqrt{3}}{2}\right)$;

P_2 has the new coordinates $\left(\frac{-4\sqrt{3} + 5}{2}, \frac{4 + 5\sqrt{3}}{2}\right)$.

Example 2. Find the equations relating coordinates in C and C' when C' is obtained from C by a rotation of (a) 45°, (b) -30°.

Solution:

(a) Since $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, we have, upon substitution in the preceding equation,

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}}(x + y)$$

$$y' = \frac{-1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}}(-x + y)$$

(b) Since $\sin(-30^\circ) = -\frac{1}{2}$ and $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$, we have

$$x' = \frac{\sqrt{3}}{2}x - \frac{1}{2}y$$

$$y' = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

We can solve for x and y in terms of x' and y' in Equation (3).

(1) $x \cos \alpha + y \sin \alpha = x'$
$-x \sin \alpha + y \cos \alpha = y'$.

(2) $x \cos^2 \alpha + y \sin \alpha \cos \alpha = x' \cos \alpha$
$x \sin^2 \alpha - y \sin \alpha \cos \alpha = -y' \sin \alpha$.

(3) Adding corresponding members, we have:

$$x \cos^2 \alpha + x \sin^2 \alpha = x' \cos \alpha - y' \sin \alpha ,$$

or $$x(\cos^2 \alpha + \sin^2 \alpha) = x' \cos \alpha - y' \sin \alpha ;$$

hence, $$x = x' \cos \alpha - y' \sin \alpha .$$

(4) By a similar process: $y = x' \sin \alpha + y' \cos \alpha$.

We shall refer to either of the pairs of equations

$$\begin{cases} x' = x \cos \alpha + y \sin \alpha \\ y' = -x \sin \alpha + y \cos \alpha \end{cases} \quad \text{or} \quad \begin{cases} x = x' \cos \alpha - y' \sin \alpha \\ y = x' \sin \alpha + y' \cos \alpha \end{cases}$$

as the equations of rotation.

Example 3. What equation represents the graph of $2x^2 + 4\sqrt{3}xy - 2y^2 = 16$ when the axes are rotated 30° ?

Solution

(1) Since $\theta = 30^\circ$, the equations of rotation are:

$$x = x' \cos \alpha - y' \sin \theta = \frac{1}{2}(\sqrt{3}x' - y')$$

$$y = x' \sin \alpha + y' \cos \theta = \frac{1}{2}(x' + \sqrt{3}y') .$$

(2) Substituting in the equation $2x^2 + 4\sqrt{3}xy - 2y^2 = 16$, and performing the indicated multiplications, we have

$$\frac{1}{2}(3x'^2 - 2\sqrt{3}x'y' + y'^2) + \sqrt{3}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2)$$
$$- \frac{1}{2}(x'^2 + 2\sqrt{3}x'y' + 3y'^2) = 16 .$$

(3) Simplifying, we have $x'^2 - y'^2 = 4$.

We recognize the graph of this equation to be a hyperbola. The graph in the $x'y'$- coordinate system can easily be drawn.

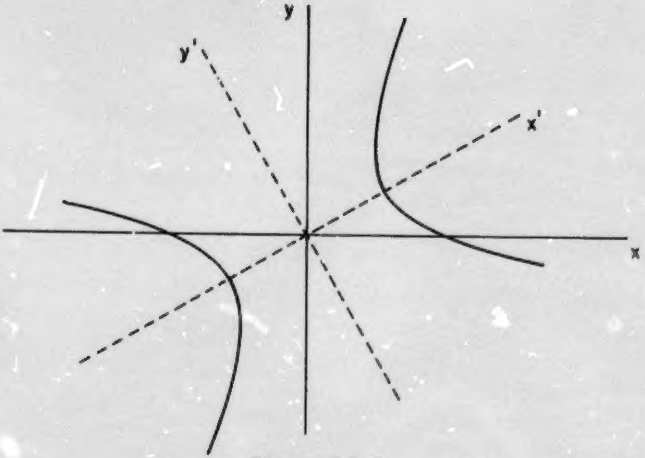


Figure 10-5

Note that a rotation of axes through an angle of 30° made the xy-term disappear. It was the elimination of the xy-term which made it possible for us to graph the curve much more readily. What we have not discussed is a method for determining through what angle a given set of axes may be rotated to eliminate the xy-term. Unfortunately we cannot develop this topic here. The interested student will enjoy studying this topic in the supplementary chapter.

Example 4. What equation represents the graph of $x^2 - y^2 = 4$ when the axes are rotated 45° ?

Solution

(1) Since $\alpha = 45^\circ$, the equations of rotation become:

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = \frac{1}{\sqrt{2}}(x' + y') .$$

(2) Substituting in the equation $x^2 - y^2 = 4$ we have

$$\frac{1}{2}(x'^2 - 2x'y' + y'^2) - \frac{1}{2}(x'^2 + 2x'y' + y'^2) = 4 .$$

(3) Simplifying, we have $x'y' = -2$.

We have here two different equations of the same equilateral hyperbola.

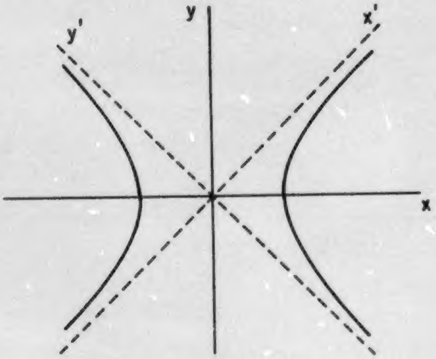


Figure 10-6

In this example, the equation with which we began had no xy-term. After a rotation, an xy-term appeared and the squared terms vanished. It may seem at first glance that we made a simple problem hard. There may be a good reason, however, why we may want to convert an equation from one form to another.

The equation $x'y' = -2$ tells us that the variables x' and y' are inversely proportional to each other. Inverse proportions are of frequent occurrence in science. For examples, in traveling a fixed distance at a constant rate the speed is inversely proportional to the time; the velocity of the wind is inversely proprotional to the spacing of the isobars (lines of constant pressure) on a weather map. We are trying to point out, in this instance, that the study of a curve whose equation has the form $xy = k$, a constant, may be more profitable than the study of the curve whose equation has the form $x^2 - y^2 = c$, a constant.

We now generalize the situation discussed in Example 4. If we start with a second degree equation containing no xy-term, a rotation of axes through an angle α , whose measure does not equal $\frac{k\pi}{2}$, for any integer k , will usually introduce an xy-term into the transformed equation. (An exception to this is the equation of a circle).

Consider the equation of the second degree which contains no xy-term.

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 ,$$

and apply the equations of rotation

$$x = x' \cos \alpha - y' \sin \alpha$$
$$y = x' \sin \alpha + y' \cos \alpha .$$

After we substitute and perform the indicated operations, this equation becomes:

$$A'x'^2 + B'x'y' + C'y'^2 + D'x + E'y + F' = 0$$

with respect to the new axes. The new constants are in terms of the constants A , C , D , E , and F . When $A' = C'$ and $B' = 0$ the equation represents a circle. (The details will be left as an exercise.)

This last equation is called the "General Equation of the Second Degree" and is written without primes as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In the Supplement to Chapter 7, we consider the method of graphing such equations. In particular you will learn how to remove the xy-term by a rotation of axes through a determined angle θ . You have already learned how to translate the origin to remove the linear terms. When both of these operations are performed, the equation of the curve is in a form which is simpler to analyze and graph.

Polar Coordinates. It was pointed out earlier that when the polar axis is rotated through an angle whose measure is α , the point $P = (r,\theta)$ will have new coordinates $(r , \theta - \alpha)$. Figure 10-4 illustrated this relation.

Let us now consider a polar equation

$$(1) \qquad r = \frac{ep}{1 - e \cos (\theta - \alpha)}$$

which represents a conic whose axis makes an angle whose measure is α with the polar axis. We illustrate an ellipse in such a position in Figure 10-7.

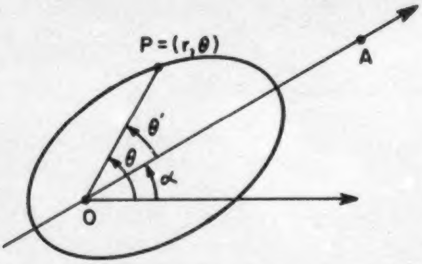


Figure 10-7

If the polar axis is now rotated through an angle whose measure is α, then an equation relative to the new polar axis, $\overrightarrow{OA}$, will be

$$r = \frac{ep}{1 - e\cos\theta'}, \text{ where } \theta' = \theta - \alpha . \tag{2}$$

You will recognize this as a polar equation of a conic with focus at the pole and axis along the new polar axis as discussed in Chapter 7.

This rotation enables us to graph the same curve by using a simpler equation. This effect was observed earlier in Section 10-3 which was concerned with rectangular coordinates.

The polar equation which represents a circle is $r = k$, a constant. This equation is independent of θ and is not changed by any change in θ.

Example. Graph $x = \dfrac{18}{3 - 2\cos(\theta + 60^\circ)}$

Solution. We first rotate the polar axis through an angle of -60°. The equation of the curve relative to the new polar axis will be

$$r = \frac{18}{3 - 2\cos\theta'} .$$

This equation represents an ellipse with its focus at the origin and with its major axis along the new polar axis as shown in Figure 10-8.

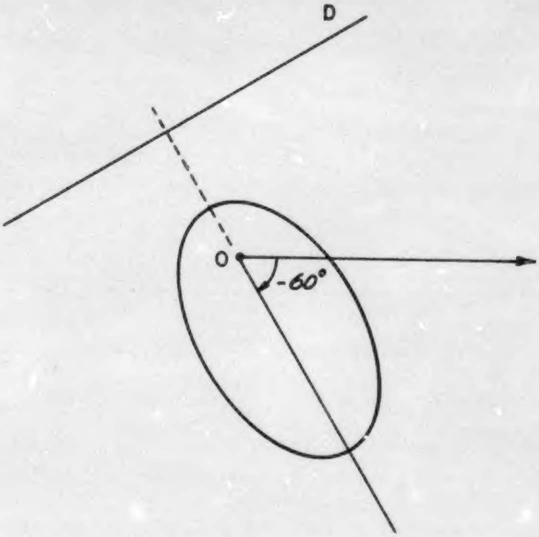


Figure 10-8

Exercises 10-3

1. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are vertices of a right triangle. Find the coordinates of these points after the axes are rotated 150°. Using the new coordinates, show that the area of the triangle has not changed.

2. What is an equation in terms of x' and y' of the line $3x + 2y - 8 = 0$ after the axes have been rotated -30°? What is the slope of this line in the new coordinate system?

3. Given the equations of rotation

$$x = x' \cos \alpha - y' \sin \alpha ,$$
$$y = x' \sin \alpha + y' \cos \alpha .$$

   Solve these equations for. x' and y'.

4. What is an equation of the parabola $x^2 = y$ with respect to axes making an angle of 45° with the original axes?

5. Find the transformed equation if the axes are rotated through the indicated angle.

   (a) $x^2 - \sqrt{3}xy + 2y^2 = 3$, $\theta = 30^\circ$

   (b) $23x^2 + 8xy + 17y^2 = 25$, θ is the angle whose tangent equals $\frac{1}{2}$.

   (c) $xy = 4$, $\theta = \frac{3\pi}{4}$

   (d) $y^2 = 4x$, $\theta = \frac{\pi}{2}$

6. Given a circle whose equation is $x^2 + y^2 = r^2$. Find the equation of this circle with respect to the new axes after the original axes undergo a rotation through any angle whose measure is α .

7. Graph each of the following after rotating the polar axis to simplify the equation.

   (a) $r = \dfrac{6}{2 - \cos(\theta - 60^\circ)}$

   (b) $r = \dfrac{10}{5 + 3\cos(\theta - 120^\circ)}$

   (c) $r = \dfrac{3}{1 + \sin(\theta - 30^\circ)}$

Challenge Problems

1. Given the general equation of the second degree $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Find an equation of its graph if the axes are rotated through an angle of θ . Let A' , B' , and C' be the coefficients of x'^2 , $x'y'$, and y'^2 respectively. Prove that $B'^2 - 4A'C' = B^2 - 4AC$. (This expression $B^2 - 4AC$ is called the characteristic of the equation.)

2. A set of axes is rotated through an angle of measure α so that the equations of rotation are:

$$\begin{cases} x = x' \cos \alpha - y' \sin \alpha \\ y = x' \sin \alpha + y' \cos \alpha \end{cases}$$

This rotation is followed by a second rotation through an angle of measure θ so that the equations of rotation are:

$$\begin{cases} x' = x'' \cos \theta - y'' \sin \theta \\ y' = x'' \sin \theta + y'' \cos \theta \end{cases}$$

Prove analytically that the coordinates (x,y) and (x'',y'') are related by:

$$\begin{cases} x = x'' \cos (\theta + \alpha) - y'' \sin (\theta + \alpha) \\ y = x'' \sin (\theta + \alpha) + y'' \cos (\theta + \alpha) \end{cases}$$

10-4. Invariant Properties.

It was mentioned in Section 10-1 that certain properties of geometric objects often remain the same under transformations. Exactly which properties remain invariant depends, of course, upon the given transformations.

The geometry we are studying, called Euclidean geometry, is identified by the fact that the measure of both distance and angle of geometric figures remain invariant under translation and rotation of axes. Many other geometric properties also remain invariant. These include the order of points on a line, collinearity of points, and concurrence of lines. Here we shall discuss only the measures of distance and angle. The other geometric properties will be illustrated in the exercises.

We shall first consider the distance between two points in a plane under a translation of axes.

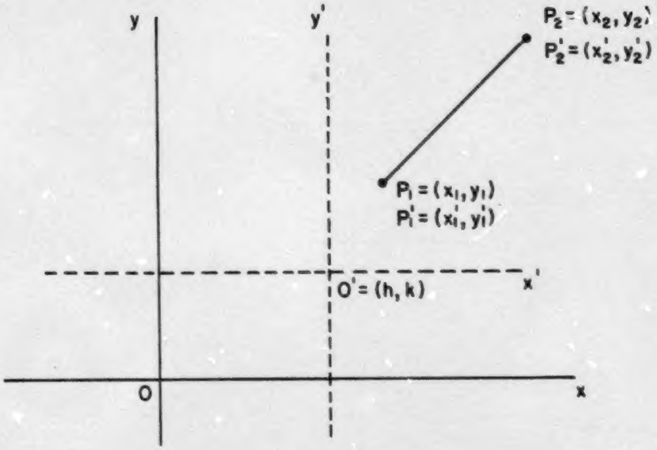


Figure 10-9

In the figure, the x-axis and y-axis with origin at O have been translated so that the new origin is at $O' = (h,k)$ with respect to the old axes. Observers at both O and O' look at the same two objects and consider the distance between them. The observer at O refers to their locations as positions P_1 and P_2, and the distance between them as s, while the observer at O' refers to the positions as P_1' and P_2' and the distance between them as s'.

You and I know that $s = s'$. But how can the two observers reconcile their observations? To answer this question, we list the known facts:

(1) $s = d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

and $s' = d(P_1',P_2') = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}$

(2) The equations of translation relating the coordinates are:

$$x' = x - h$$
$$y' = y - k$$

Using these facts, we have:

(3) $\begin{cases} x_2' = x_2 - h \\ x_1' = x_1 - h . \end{cases}$ Therefore $x_2' - x_1' = x_2 - x_1$,

and $\begin{cases} y_2' = y_2 - k \\ y_1' = y_1 - k . \end{cases}$ Therefore, $y_2' - y_1' = y_2 - y_1$

(4) We substitute the expressions from (3) in the formula for s', obtaining $s' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ which is identical with the formula for s, as was to be proved.

A numerical problem may help in making the above discussion clearer. Let $P_1 = (4,6)$, $P_2 = (-1,2)$ and $O' = (5,-2)$. Thus the equations of translation are $x' = x - 5$ and $y' = y + 2$.

The coordinates of P_1', are $(-1,8)$ and of P_2' are $(-6,4)$. Thus $d(P_1',P_2') = \sqrt{25 + 16} = \sqrt{41}$, $d(P_1,P_2) = \sqrt{25 + 16} = \sqrt{41}$, and we have $d(P_1,P_2) = d(P_1',P_2')$.

What if the axes in the above problem had been rotated instead of translated? We would then consider the following:

(1) The equations of rotation are:

$$\begin{cases} x' = x \cos \theta + y \sin \theta , \\ y' = -x \sin \theta + y \cos \theta . \end{cases}$$

so that $\begin{cases} x_1' = x_1 \cos \theta + y_1 \sin \theta \text{ and } x_2' = x_2 \cos \theta + y_2 \sin \theta ; \\ y_1' = -x_1 \sin \theta + y_1 \cos \theta \text{ and } y_2' = -x_2 \sin \theta + y_2 \cos \theta . \end{cases}$

Therefore, $x_2' - x_1' = (x_2 - x_1) \cos \theta + (y_2 - y_1) \sin \theta$, and

$$y_2' - y_1' = -(x_2 - x_1) \sin \theta + (y_2 - y_1) \cos \theta .$$

(2) Squaring and adding corresponding members, we have:

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 = (x_2 - x_1)^2(\cos^2 \theta + \sin^2 \theta)$$
$$+ (y_2 - y_1)^2(\cos^2 \theta + \sin^2 \theta) ,$$

or $(x_2' - x_1')^2 + (y_2' - y_1')^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

(3) Thus $d(P_1',P_2') = d(P_1,P_2)$.

We see that distance is invariant under both rotation and translation of axes and we state this as a theorem:

THEOREM 10-2. The measure of distance between two points is invariant under:

(a) a translation of axes

(b) a rotation of axes.

The invariance of the measure of angle under a translation or a rotation of axes follows directly from Theorem 10-2.

Consider $\triangle ABC$ determined by $A = (x_1,y_1)$, $B = (x_2,y_2)$, and $C = (x_3,y_3)$. Under either one of the above transformations, the points A, B, and C will have new coordinates. They will now be designated as $A' = (x_1',y_1')$, $B' = (x_2',y_2')$, and $C' = (x_3',y_3')$ with respect to the new axes.

Since distance between points is invariant, we have $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\overline{AC} \cong \overline{A'C'}$. Hence, $\triangle ABC \cong \triangle A'B'C'$ and the corresponding angles are congruent.

THEOREM 10-3. The measure of angle is invariant under:

(a) a translation of axes.

(b) a rotation of axes.

It would have been possible to prove the invariance of the measure of angle under translation or rotation independently of the invariance of distance discussed here. We could start with the formula

$$\cos\theta = \frac{1 + m_1m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}},$$

and consider the lines $L_1 : ax + by + c = 0$ and $L_2 : a'x + b'y + c' = 0$.

Upon the translation of axes, the lines L_1 and L_2, with respect to the new axes have the equations

$$L_1' : a(x' + h) + b(y' + k) + c = 0,$$

or

$$L_1' : ax' + by' + (ah + bk + c) = 0.$$

and

$$L_2' : a'(x' + h) + b'(y' + k) + c' = 0,$$

$$L_2' : a'x' + b'y' + (a'h + h'k + c') = 0.$$

The slope of L_1' is given by $m_1' = \frac{a}{b} = m_1$, and the slope of L_2' is given by $m_2' = -\frac{a'}{b'} = m_2$. Since the slopes are equal, $\cos \theta' = \cos \theta$ and $\theta' = \theta$ for the principal value. Hence the measure of angle is invariant under translation.

The proof of the invariance of angle under rotation involves considerable algebraic manipulation and is left as a "challenge" exercise.

Exercises 10-4

1. (a) Find an equation of the line through $A = (2,1)$ and $B = (0,4)$ and draw the line.

   (b) Find the coordinates of A and B and an equation of the line after the origin has been translated to $(-4,-6)$.

   (c) Verify that $d(A,B)$ is invariant under this translation.

2. (Refer to Exercise 1 above)

   (a) Find the coordinates of A and B and an equation of the line after the axes have been rotated 90° .

   (b) Verify that $d(A,B)$ is invariant under this rotation.

3. Given line $L : 4x - 3y - 12 = 0$ passing through $A = (0,-4)$, $B = (2,-\frac{4}{3})$ and $C = (3,0)$.

   (a) Find the coordinates of these points (now renamed A' , B' , and C' respectively) and an equation of the line (now called L') when the origin has been translated to $(-1,-1)$.

   (b) Verify that the order of points A' , B' , and C' is the same as that of A , B , and C . (That is, order of points on a line is invariant.

   (c) Verify that A' , B' , and C' are collinear. (That is, collinearity of points is invariant under translation.)

4. Given lines $L_1 : 4x - 3y - 5 = 0$, $L_2 : x - 2y = 0$, and $L_3 : 5x - 3y - 7 = 0$.

   (a) Verify that L_1 , L_2 , and L_3 are concurrent.

   (b) Find equations of these lines (now renamed L_1' , L_2' and L_3') after the origin has been translated to $(3,-2)$.

(c) Verify that L_1', L_2' and L_3' are concurrent. (That is, concurrence of lines is invariant under translation.)

(d) What is the relation between the point of concurrency of L_1, L_2 and L_3 and that of L_1', L_2' and L_3' ?

(e) Do parts (b), (c), (d) if, instead of translating the origin, the axes are rotated 45°.

5. Given lines $L_1 : 3x + 2y - 8 = 0$
and $L_2 : 5x - y - 9 = 0$.

(a) Find the acute angle between L_1 and L_2 at their point of intersection.

(b) Find equations of L_1 and L_2 (now called L_1' and L_2') after the origin is translated to $(2,2)$.

(c) Find the angle between L_1' and L_2' and verify that the angle is invariant under translation.

Challenge Problem

Prove that the measure of angle is invariant under a rotation of axes, without making use of the invariance of distance.

10-5. Point Transformations

In the previous sections we considered an operation called the "transformation of axes". We now consider another type of tranformation which achieves similar results from a different point of view. However, this new point of view leads to significant results, such as the transformation of a given curve into a corresponding curve which is not congruent to the original. This we could not achieve by the original approach.

We now consider a transformation, called a point transformation, which carries each point A into another point A' in the same plane. Thus the points of a figure F are carried into a set of points forming a figure F', as shown in Figure 10-10. The axes remain fixed.

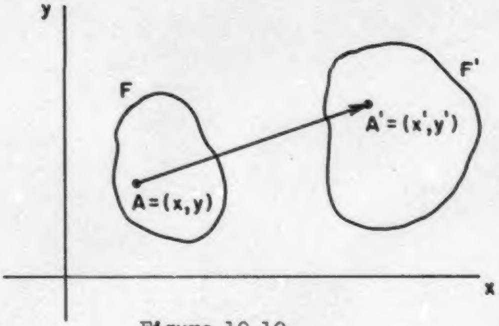


Figure 10-10

In this sense a transformation is an operation by which each element of a geometric figure is replaced by another element. Another way of expressing this concept is that a transformation is a one-to-one correspondence or mapping of each point of A onto a corresponding point A' . The plane is mapped onto itself. A point transformation is written symbolically as A ⟶ A' and A' is called the image of A .

We can also consider translations and rotations as point transformations. In Figure 10-11, $P = (x,y)$ has been mapped into $P' = (x',y')$ by "moving" the point horizontally a distance of h and vertically a distance of k . Thus

$$\begin{cases} x' = x + h \\ y' = y + k \ . \end{cases}$$

Another way to write this transformation is $(x,y) \longrightarrow (x + h, y + k)$. This form will be used frequently in the remainder of the text.

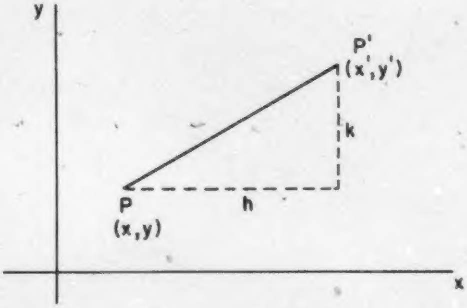


Figure 10-11

This pair of equations is similar to those derived earlier for a translation of axes; they differ only in the signs of h and k . This occurs because we are now moving the point and keeping the axes fixed.

The following example will illustrate this fact.

Let points $A = (2,0)$, $B = (2,1)$ and $C = (4,1)$ be the vertices of a triangle as shown in Figure 10-12. These points now undergo a point transformation given by

$$\begin{cases} x' = x + 4 \\ y' = y + 6 . \end{cases}$$

Thus

$$A = (2,0) \longrightarrow A' = (6,6)$$
$$B = (2,1) \longrightarrow B' = (6,7)$$
$$C = (4,1) \longrightarrow C' = (8,7)$$

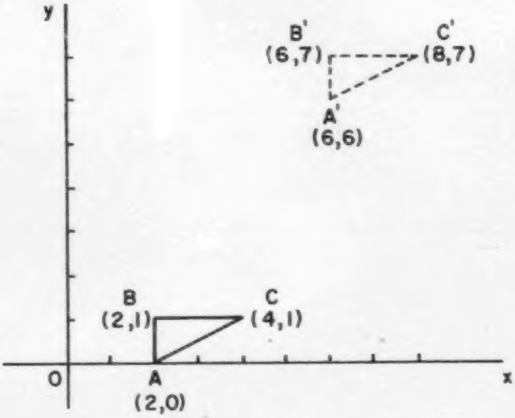


Figure 10-12

You will note that $\triangle ABC$ has been mapped into $\triangle A'B'C'$. You should also observe that the same "visual effect" could have been achieved by translating the x- and y-axes to a new origin at $(-4,-6)$. What we are saying is that $\triangle ABC$ would have the same relative position and appearance to a person standing at point $(0,0)$ as $\triangle A'B'C'$ would have to a person standing at point $(-4,-6)$. Note that the coordinates $(-4,-6)$ are the negatives of the values of h and k used in the point transformation.

A rotation is now considered as a mapping in which each point in the plane is mapped onto a point the same distance from the origin as previously. When $P \longrightarrow P'$ and $Q \longrightarrow Q'$, the rotation will map $\angle POP'$ into the congruent angle QOQ' . In the figure, $A = (2,0)$ has been mapped onto

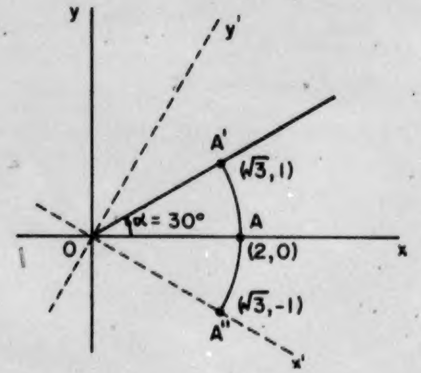


Figure 10-13

$A' = (\sqrt{3},1)$ by rotating through an angle whose measure is 30° ; both points are at a fixed distance of two units from O . A comparable visual effect would have been achieved if the axes had been rotated through an angle whose measure is -30° , and $A'' = (\sqrt{3},-1)$ located on the x'-axis. The idea we are emphasizing is that A has the same relative position to an observer at A" as A' has to an observer at A . Also, $\overline{OA}$ has the same position with respect to the x'- and y'-axes as $\overline{OA'}$ has with respect to the x- and y-axes. A similar statement could be made regarding the rotation of any polygon or for any general figure F . The angle of rotation could be generalized to be any angle whose measure is α .

We now return to the concept of reflection which was discussed in detail in Section 6-2 with relation to the symmetry of curves. We shall now define certain relfections in terms of point transformation as follows:

(1) A reflection with respect to the x-axis is given by $(x,y) \rightarrow (x,-y)$

(2) A reflection with respect to the y-axis is given by $(x,y) \rightarrow (-x,y)$

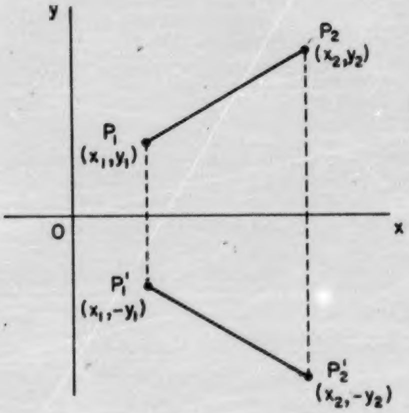
(3) A reflection with respect to the origin is given by $(x,y) \rightarrow (-x,-y)$.

Note our use here of the alternate notation indicated earlier in this section.

Reflections with respect to lines L and L' parallel to the x- and y-axes respectively are best treated by translating the x- and y-axes to coincide with L and L' . In accordance with our practice regarding notation we shall now refer to lines L and L' as the x'- and y'-axes respectively. Thus the point transformations are considered with respect to the x'- and y'-axes and to the new origin at $O' = (h,k)$ as shown in Figure 10-14d.

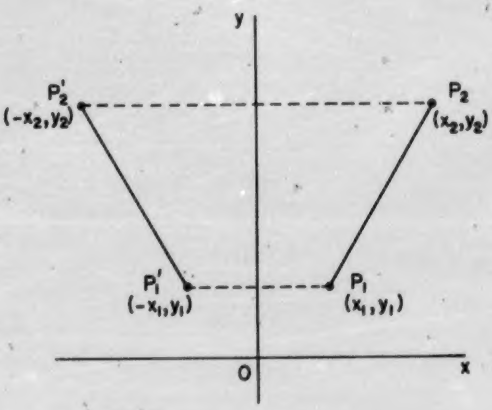
We can consider reflections with respect to any point or line but the equations of transformation are often difficult to state explicitly. We consider this subject beyond the scope of this text and refer you to the challenge exercises in Section 6-2.

Sone reflections of segments are indicated in Figure 10-14.


y
P_2
(x_2,y_2)
P_1
(x_1,y_1)
0
x
P_1'
$(x_1,-y_1)$
P_2'
$(x_2,-y_2)$

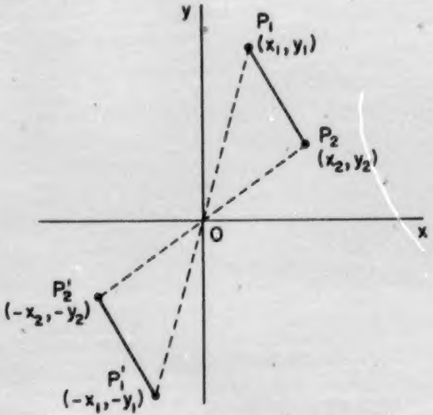
$(x,y) \longrightarrow (x,-y)$

Figure 10-14a


y
P_2'
$(-x_2,y_2)$
P_2
(x_2,y_2)
P_1'
$(-x_1,y_1)$
P_1
(x_1,y_1)
0
x

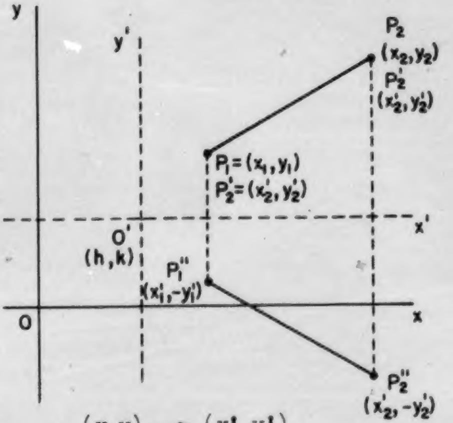
$(x,y) \longrightarrow (-x,y)$

Figure 10-14b


y
P_1
(x_1,y_1)
P_2
(x_2,y_2)
0
x
P_2'
$(-x_2,-y_2)$
P_1'
$(-x_1,-y_1)$

$(x,y) \longrightarrow (-x,-y)$

Figure 10-14c


y
y'
P_2
(x_2,y_2)
P_2'
(x_2',y_2')
$P_1=(x_1,y_1)$
$P_2'=(x_2',y_2')$
x'
O'
(h,k)
P_1''
$(x_1',-y_1')$
0
x
P_2''
$(x_2',-y_2')$

$(x,y) \longrightarrow (x',y')$

$(x',y) \longrightarrow (x',-y')$

Figure 10-14d

In each of the above illustrations, $d(P_1,P_2) = d(P_1',P_2')$. It is possible to prove that distance is invariant under the set of all reflections. We present here a proof of the first case where a line segment is reflected with respect to the x-axis.

Referring to Figure 10-14a, we have $d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and $d(P_1',P_2') = \sqrt{(x_2 - x_1)^2 + (-y_2 + y_1)^2}$. Since $(-y_2 + y_1)^2 = (y_2 - y_1)^2$ we have $d(P_1,P_2) = d(P_1',P_2')$.

It is also possible to prove that any translation, rotation, or combination of translations and rotations, can be accomplished by a series of no more than three line reflections. A proof will be found in the Supplement to Chapter 10. We shall merely illustrate it here in three examples.

Example 1. Show how the translation of $\triangle ABC$ to the new position indicated by $\triangle A''B''C''$ can be effected by a series of line reflections.

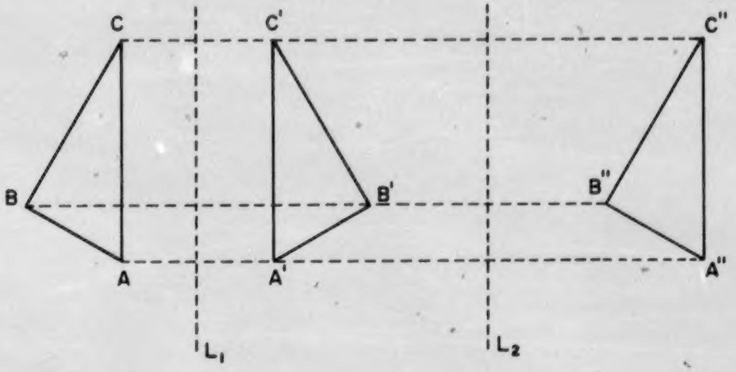


Figure 10-15

In Figure 10-15, we see that $\triangle ABC$ has been translated to $\triangle A''B''C''$ by a series of two refelctions. The axes of reflection, L_1 and L_2 , were selected parallel to $\overline{AC}$. Axis L_1 may be chosen freely but there is only one position possible for L_2 .

Example 2. (Same as Example 1.)

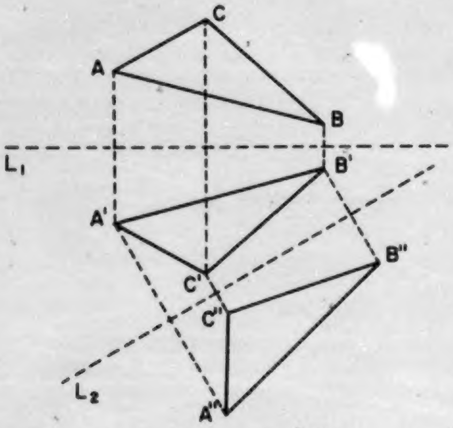


Figure 10-16

In Figure 10-16, we observe that $\triangle ABC$ has been reflected with respect to axes L_1 and L_2, with the result that it has been both translated and rotated.

Example 3. Demonstrate how axes of reflection can be selected to move a directed line segment from one position to another given position.

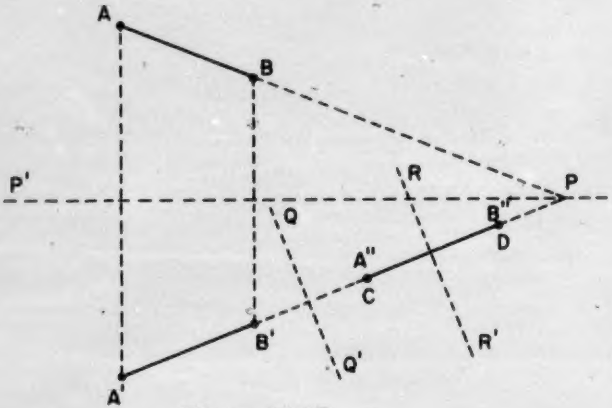


Figure 10-17

In Figure 10-17, $\overline{AB} \longrightarrow \overline{A''B''}$ by a series of at most three line reflections by using the following procedure.

(1) Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{A''B''}$ intersecting at P .

(2) Bisect angle P and call the bisector $\overleftrightarrow{PP'}$.

(3) Reflect $\overline{AB}$ with respect to $\overleftrightarrow{PP'}$. $\overline{A'B'}$, the image of $\overline{AB}$, will lie on $\overleftrightarrow{A''B''}$.

(4) Construct $\overleftrightarrow{QQ'}$, a perpendicular to $\overline{B'A''}$. Reflect $\overline{A'B'}$ with respect to $\overleftrightarrow{QQ'}$. Its image $\overline{DC}$ lies on $\overleftrightarrow{A'B'}$ and coincides with B"A" .

(5) Construct $\overleftrightarrow{RR'}$, the perpendicular bisector of $\overline{CD}$. Reflect $\overline{CD}$ with respect to $\overleftrightarrow{RR'}$. Thus $D \rightarrow A''$ and $C \rightarrow B''$ and the order of points on $\overline{A''B''}$ is the same as that of $\overline{AB}$.

The selection of axes of reflection when $\overline{AB} || \overline{A''B''}$ is left as an exercise.

The effect of one or more reflections upon a geometric figure can be studied analytically as well as by actual construction and observation. To illustrate this approach, we shall consider the point reflection $(x,y) \rightarrow (-x,-y)$.

Upon applying this transformation to the line $L : ax + by + c = 0$, the equation becomes $L' : -ax - by + c = 0$ or $ax + by - c = 0$. The lines L and L' are parallel but the intercepts on the axes have different signs. Specifically, the line $2x + 3y - 6 = 0$, with intercepts (3,0) and (0,2) transforms to the line $2x + 3y + 6 = 0$ with intercepts (-3,0) and (0,-2)

When the same transformation is applied to the circle $x^2 + y^2 = r^2$, we note that there is no change in the equation. This result verifies the fact that this circle is symmetric with respect to the origin. A similar result is obtained for the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the hyperbolas $b^2x^2 - a^2y^2 = a^2b^2$ and $xy = k$, the cubic parabola $y = x^3$, and any other curves that are symmetric to the origin.

The circle $x^2 + y^2 + Dx + Ey + F = 0$ transforms into another circle $x^2 + y^2 - Dx - Ey + F = 0$. The radii have the same measure but the center is now at $(\frac{D}{2}, \frac{E}{2})$ instead of at $(-\frac{D}{2}, -\frac{E}{2})$. Figure 10-16 illustrates the effect of the point reflection $(x,y) \rightarrow (x'y')$ upon the circle $C : x^2 + y^2 - 4x - 6y - 12 = 0$. The equation of the transformed circle is $C' : x^2 + y^2 + 4x + 6y - 12 = 0$. C and C' both have a radius of 5 but the center of C' is at (-2,-3) while that of C is at (2,3) .

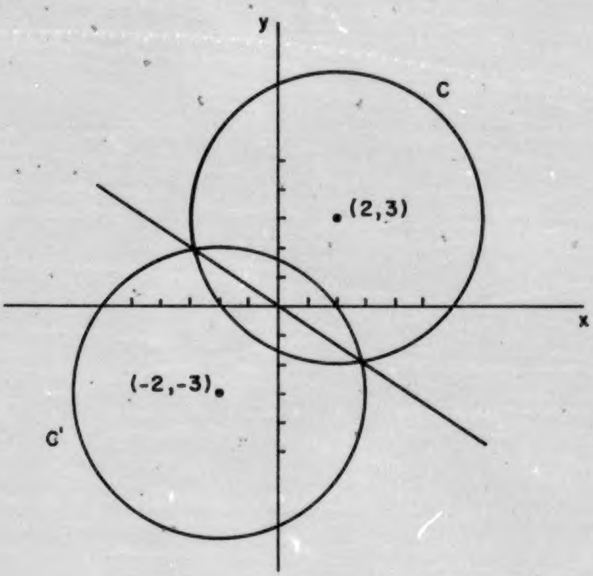


Figure 10-18

A second reflection $(x',y') \longrightarrow (x'',y'')$ with respect to the same point will map C' into $C'' : x^2 + y^2 - 4x - 6y - 12 = 0$ and we observe that $C'' = C$. A similar result is obtained when any reflection is followed by one of the same type and with respect to the same point or line. A number of transformations, other than reflections have this same property. We shall discuss one of these in the next section.

A variety of point transformations will be presented in the exercises.

Exercises 10-5

1. Given points $A = (1,2)$ and $B = (3,-4)$. Reflect A and B with respect to the

   (a) x-axis
   (b) y-axis
   (c) origin
   (d) line $x = 6$.

   Verify in each case that $d(A,B)$ is invariant.

2. The equation $x' = x + 2$ may represent a point transformation along the x-axis. Select any three points on the x-axis, find their images under the transformation, and determine two properties which remain invariant.

3. Perform Exercise 2 for the transformation $x' = 2x$. Find three properties invariant under this transformation.

4. Show that the angle between the lines $L_1 : y = 0$ and $L_2 : y = x$ is preserved under rotation through an angle of measure $\frac{\pi}{4}$.

5. Show the effect of the mapping indicated for each of the following curves by graphing both the original curve and its image on the same set of axes.

   (a) $y^2 = x$; $(x,y) \longrightarrow (-x,y)$

   (b) $x^2 = y$; $(x,y) \longrightarrow (-x,-y)$

   (c) $xy = 6$; $(x,y) \longrightarrow (-x,-y)$

   (d) $4x^2 - 9y^2 = 36$; $(x,y) \longrightarrow (3x,2y)$

   (e) $x^2 + y^2 - 2x + 4y + 4 = 0$; $(x,y) \longrightarrow (-x,y)$

   (f) $y = x^3$; $(x,y) \longrightarrow (x,-y)$

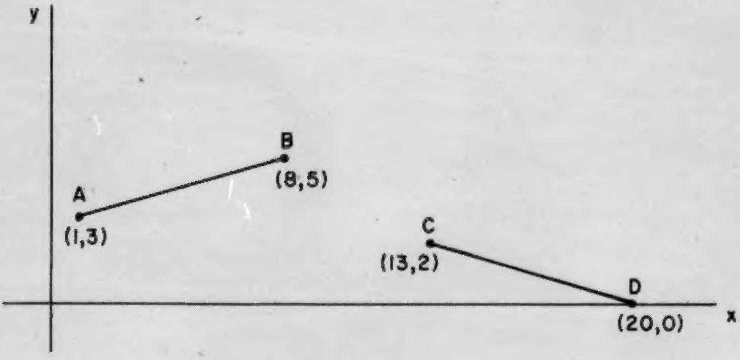
   (g) $y = \sin x$; $(x,y) \longrightarrow (x,-y)$

   (h) $y = \tan x$; $(x,y) \longrightarrow (-x,y)$

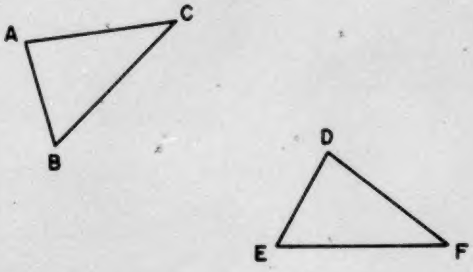
   (i) $y = 2^x$; $(x,y) \longrightarrow (-x,y)$

6. $A = (-2,1)$, $B = (5,-2)$, and $C = (3,3)$ are vertices of a triangle. They are rotated about the origin through an acute angle θ such that $\tan \theta = \frac{3}{4}$. Test and verify three properties which remain invariant under this rotation.

7. (a) Given the segments $\overline{AB}$ and $\overline{CD}$ as shown in the figure. Show, by construction, how $\overline{AB}$ can be mapped into $\overline{CD}$ by means of line reflections.



(b) Trace congruent triangles ABC and DEF keeping their relative positions. Show how to map $\triangle ABC$ into $\triangle DEF$ by the method used in part (a).



8. The points on the following curves are rotated through an angle of measure $\frac{\pi}{6}$ with respect to the origin. Find the equations of the transformed curves. Sketch each of the curves and its image on the same set of axes.

   (a) $3x + 2y - 8 = 0$

   (b) $x^2 + y^2 = 25$

   (c) $y^2 = 4x$

9. Discuss the transformation $(x,y) \longrightarrow (-y + 3, x + 1)$ by finding the images of the curves in Exercise 8.

10. Determine whether parallelism is preserved when the lines $L_1 : 3x - 2y + 5 = 0$ and $L_2 : 3x - 2y - 3 = 0$ undergo the mapping $(x,y) \longrightarrow (x + y, 2x - y)$.

10-6. Inversions.

We conclude with a discussion of a point transformation called an inversion.

Consider a circle C with radius r and center at O. Select any point $P \neq O$, $d(O,P) \geq \frac{1}{2}r$, and draw $\overleftrightarrow{OP}$. With P as a center and $\overline{OP}$

as radius draw an arc intersecting C at R . Finally, with R as center and a radius r draw an arc intersecting $\overleftrightarrow{OP}$ in P' . The construction is shown in Figure 10-19 . (Note that this construction requires that the circle be intersected at point R .)

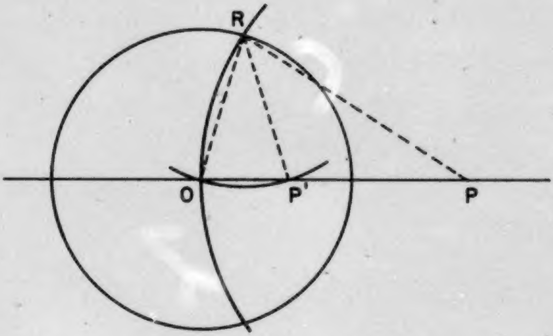


Figure 10-19

$\triangle ORP$ is isosceles since $\overline{OP} \cong \overline{RP}$; $\triangle ORP'$ is isosceles since $\overline{OR} \cong \overline{RP'}$. Thus $\angle ORP \cong \angle POR \cong \angle OP'R$ and $\triangle RPO \cong \triangle ORP'$. Then $\frac{d(O,P)}{d(O,R)} = \frac{d(O,R)}{d(O,P')}$ and $d(O,P) \cdot d(O,P') = r^2$. Two points P and P' which meet this condition are said to be *mutually inverse* points with respect to circle C .

When $d(O,P) < \frac{1}{2}r$, the arc drawn with P as a center and $\overline{OP}$ as radius will not intersect the circle. In this case, construct the perpendicular bisector of $\overline{OP}$ intersecting the circle at R and $\overline{OP}$ in S . At R , construct $\angle ORT \cong \angle POR$. Then $\overrightarrow{RT}$ will intersect $\overleftrightarrow{OP}$ in P' . It is left as an exercise to prove that $\overline{OP} \cdot \overline{OP'} = r^2$.

DEFINITION. An *inversion* is a point transformation which maps each of two arbitrary points which are mutually inverse into the other.

Circle C is called the *circle of inversion* and point O is called the *center of inversion*. Point P' is said to be the inverse or image of P , and vice-versa.

Each point on the unit circle is its own image; each point outside this circle has a unique image inside; and, with the exception of the origin, each point inside the circle has a unique image outside. This is true because if $d(O,P) < r$, we have $d(O,P') > r$, and for $d(O,P) > r$, we have $d(O,P') < r$. For any point on the unit circle, $d(O,P) = d(O,P') = r$.

We now obtain an analytic representation for such a transformation. For simplicity, we let $r = 1$.

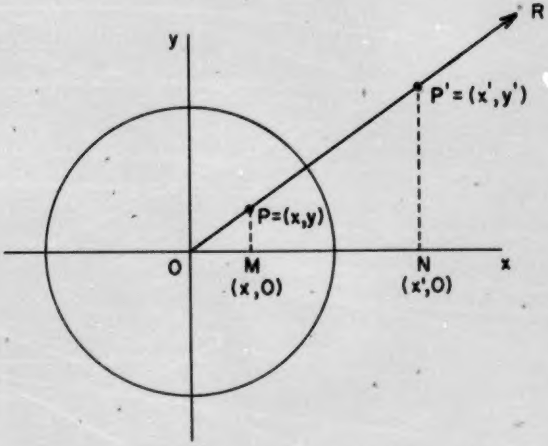


Figure 10-20

Given a unit circle C with its center at the origin. Draw any ray $\overrightarrow{OR}$ and locate on $\overrightarrow{OR}$ mutually inverse points P and P'. Construct perpendiculars from P and P' to the x-axis, intersecting the axis at M and N respectively.

(1) Since $\triangle OMP \cong \triangle ONP'$, $\frac{d(O,P)}{d(O,P')} = \frac{x}{x'}$.

(2) By definition, we have $d(O,P) \cdot d(O,P') = 1$ or $d(O,P) = \frac{1}{d(O,P')}$.

(3) Thus by substitution, $\frac{x}{x'} = \frac{1}{(d(O,P'))^2} = (d(O,P))^2$.

(4) Since $(d(O,P))^2 = x^2 + y^2$ and $(d(O,P'))^2 = x'^2 + y'^2$, we have $\frac{x}{x'} = \frac{1}{x'^2 + y'^2}$ and $\frac{x}{x'} = x^2 + y^2$.

(5) Thus $x = \frac{x'}{x'^2 + y'^2}$ and $x' = \frac{x}{x^2 + y^2}$.

(6) In a similar fashion, $y = \dfrac{y'}{x'^2 + y'^2}$ and $y' = \dfrac{y}{x^2 + y^2}$.

(7) The pairs of equations:

$$\begin{cases} x = \dfrac{x'}{x'^2 + y'^2} \\ y = \dfrac{y'}{x'^2 + y'^2} \end{cases} \quad \text{and} \quad \begin{cases} x' = \dfrac{x}{x^2 + y^2} \\ y' = \dfrac{y}{x^2 + y^2} \end{cases}$$

are called the equations of the inversion transformation. We shall now investigate the effect of applying this transformation to several curves.

Example 1. What is the inverse of a straight line with respect to a unit circle?

(1) Let $L : ax + by + c = 0$ with $c \neq 0$. Then L' , the inverse of L , has the equation

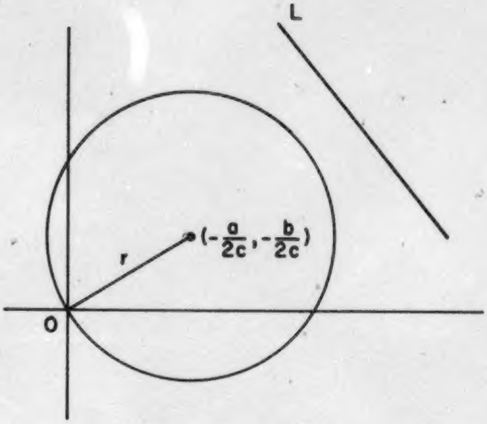
$$\frac{ax'}{x'^2 + y'^2} + \frac{by'}{x'^2 + y'^2} + c = 0 .$$

(2) Thus $c(x'^2 + y'^2) + ax' + by' = 0$, or $x'^2 + y'^2 + \frac{a}{c}x' + \frac{b}{c}y' = 0$.

(3) Completing the squares, we have:

$$(x' + \frac{a}{2c})^2 + (y' + \frac{b}{2c})^2 = \frac{a^2 + b^2}{4c^2}$$

and we recognize the graph of a circle with center at $(-\frac{a}{2c} , -\frac{b}{2c})$, with $r = \dfrac{\sqrt{a^2 + b^2}}{2c}$, and passing through the origin as illustrated in Figure 10-21 .



$$r = \frac{\sqrt{a^2 + b^2}}{2c}$$

Figure 10-21

Thus a line not passing through the origin transforms into a circle passing through the origin. The converse of this theorem is also true: a circle passing thru the origin transforms into a straight line not passing through the origin. The proof is left as a Challenge Problem.

There is an interesting special case of this problem. Note that in the example given we defined the line L by $ax + by + c = 0$ and $c \neq 0$. What if $c = 0$?

In this case, we have $L : ax + by = 0$ or $y = -\frac{a}{b}x$ or $y = mx$ where m is the slope. The inversion transformation yields

$$\frac{y'}{x'^2 + y'^2} = \frac{mx'}{x'^2 + y'^2} .$$

Thus $y' = mx'$ and we observe that a line passing through the origin transforms into itself. Another way of saying this is that a line passing through the origin remains invariant under an inversion transformation.

Example 2. What is the inverse, with respect to the unit circle, of a network of lines $x = c$, parallel to the y-axis, and $y = k$, parallel to the x-axis?

(1) The lines $x = c$ transform into

$$\frac{x'}{x'^2 + y'^2} = c \quad \text{or} \quad c(x'^2 + y'^2) = x'$$

(2) Thus $x'^2 + y'^2 - \frac{x'}{c} = 0$, or $(x' - \frac{1}{2c})^2 + y'^2 = \frac{1}{4c^2}$.

This equation represents a whole "family of circles" passing through the origin with centers at $(\frac{1}{2c}, 0)$.

(3) In a similar fashion, the lines $y = k$ transform into a family of circles with centers at $(0, \frac{1}{2k})$ and passing through the origin.

A part of a network of lines and the circles which are their inverses are shown in Figure 10-22.

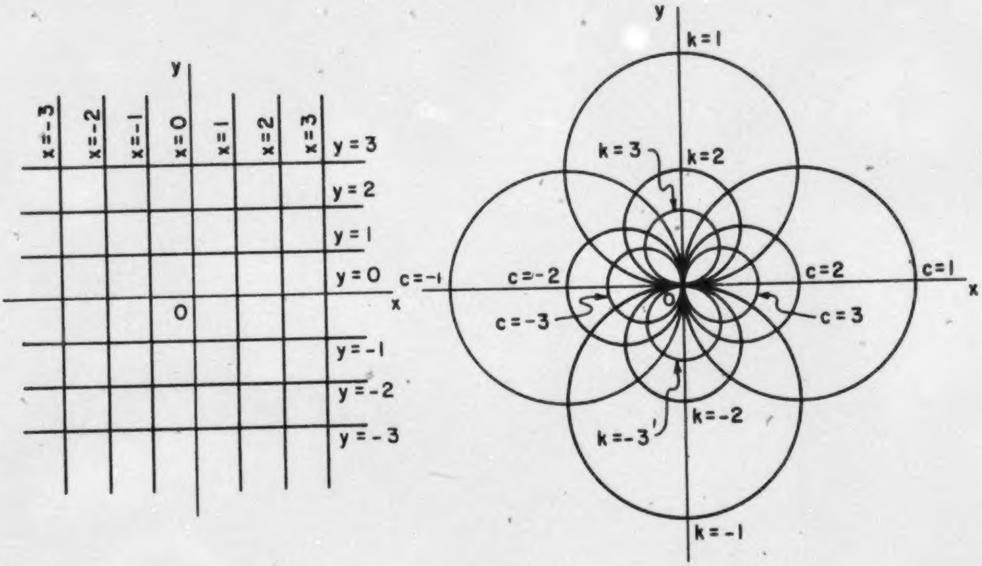

y
x=-3
x=-2
x=-1
x=0
x=1
x=2
x=3
y=3
y=2
y=1
y=0
x
O
y=-1
y=-2
y=-3
y
k=1
k=3
k=2
c=-1
c=-2
c=2
c=1
x
O
c=-3
c=3
k=-2
k=-3
k=-1

Figure 10-22

You have already observed an unusual result. For the first time in this discussion, a curve has been transformed into a different curve. Such an event was made possible because we are dealing with point transformations. In Figure 10-20, a different scale was used for the two drawings.

As a final example, we consider the following;

Example 3. What is the inverse of a circle with respect to the unit circle?

(1) Consider the general equation of a circle

$C : x^2 + y^2 + Dx + Ey + F = 0$, and apply the equations of inversion. Thus we have

$$\frac{x'^2}{(x'^2+y'^2)^2} + \frac{y'^2}{(x'^2+y'^2)^2} + \frac{Dx'}{x'^2+y'^2} + \frac{Ey'}{x'^2 \quad y'^2} + F = 0$$

or $$\frac{1}{x'^2 + y'^2} + \frac{Dx'}{x'^2 + y'^2} + \frac{Ey'}{x'^2 + y'^2} + F = 0$$

(2) Thus since $x'^2 + y'^2 \neq 0$, $F(x'^2 + y'^2) + Dx' + Ey' + 1 = 0$

or $x'^2 + y'^2 + \frac{D}{F}x' + \frac{E}{F}y' + \frac{1}{F} = 0$

(3) Substituting $D' = \frac{D}{F}$, $E' = \frac{E}{F}$, $F' = \frac{1}{F}$, we get

$$C' : x'^2 + y'^2 + D'x' + E'y' + F' = 0$$

which we recognize as a different circle (in general).

It may be of interest to discover whether C and C' are related to each other in any way.

Exercises 10-6

The first five exercises are concerned with the effect of inverting the given curve with respect to the unit circle. The equations of the inversion are

$$x = \frac{x'}{x'^2 + y'^2} , \; y = \frac{y'}{x'^2 + y'^2} .$$

For each exercise, draw the circle of inversion, the original curve, and its inverse on the same graph.

1. $3x + 2y - 6 = 0$

2. $y = 5x$

3. $y = 3$

4. $y^2 = 4x$ (The graph of the inverted curve is optional)

5. $(x - 4)^2 + (y - 4)^2 = 16$

6. Find the inverse of each of the following lines with respect to the unit circle. Graph all of them on one set of axes and all their inverses on another set. The lines are: $x = \pm 2$, $x = \pm 4$, $x = \pm 6$, $y = \pm 2$, $y = \pm 4$, and $y = \pm 6$.

7. In Exercise 1 you found the inverse of the line $L : 3x + 2y - 6 = 0$. Call the inverse L' . Now apply the same transformation to L' . What can you conjecture from the result?

8. Derive equations of inversion with respect to a circle whose radius is r and center at the origin.

9. The following four points are collinear: $A = (0,-3)$, $B = (1,-1)$, $C = (2,1)$ and $D = (3,3)$. Find the inverse of each of these points with respect to the circle $x^2 + y^2 = 4$ and call the inverse points A' , B' , C' , and D' . Prove that

$$\frac{\dfrac{d(A,C)}{d(A,D)}}{\dfrac{d(B,C)}{d(B,D)}} = \frac{\dfrac{d(A',C')}{d(A',D')}}{\dfrac{d(B',C')}{d(B',D')}}$$

(This ratio is called a cross-ratio in more advanced geometries).

10. Refer to the text and perform the construction of the inverse point P' when $r < \frac{1}{2}$. Prove that $\overline{OP} \cdot \overline{OP'} = r^2$.

Challenge Problem

Prove that a circle passing through the origin inverts into a straight line not passing through the origin.

10-7. Summary and Review Exercises.

We have considered two types of geometric transformations. The first type considered a transformation as an operation which changed one set of axes into another by means of translation or a rotation or both. In a translation, the axes are shifted in such a way that they remain parallel to their original positions and oriented in the same direction; the origin is moved. In contrast, a rotation keeps the origin fixed but new axes are obtained by rotating the axes through a fixed angle. Sets of equations were derived to effect these operations. We demonstrated how a relatively complex equation could be reduced to a simpler form which then could be drawn more readily.

As a second type of transformation we considered the mapping of the plane onto itself. Rules were given by which any point or sets of points in the plane can be moved from one position to another. This set of transformations can effect translations and rotations. It can also effect reflections, inversions, and other changes. Reflections are related to the concept of symmetry in figures. Inversions can convert one type of curve into another. The exercises illustrated some other types of point transformations.

One of the principal reasons for studying transformations is to discover which geometric properties remain invariant under the stated operations. Geometries are classified on the basis of these properties. Euclidean geometry is characterized by the fact that the measures of distance and angle are invariant under the set of all rotations and translations. This set is often referred to as the set of rigid motions since those transformations preserve size and shape. Other invariant properties were considered in the exercises.

Review Exercises

PART II

The "Review Exercises" are concerned primarily with several transformations not discussed in the text. They are presented so that you may discover some significant facts for yourself and may widen your experience with the subject.

1. Find the curve into which the parabola $x^2 = 2y$ is transformed by each of the following mappings:

   (a) $(x,y) \longrightarrow (2x,3y)$
   (b) $(x,y) \longrightarrow (x + 2, 3y)$
   (c) $(x,y) \longrightarrow (x - 1, y + 2)$.

   Draw the original curve and its image for each. Can you find any invariant properties under any of these transformations?

2. The mapping $(x,y) \longrightarrow (kx,ky)$ is called the transformation of similitude. Let $k = 2$ and find the effect of this transformation upon the graphs of the following:

   (a) $2x + 3y - 6 = 0$

   (b) $x^2 + y^2 = 25$

   (c) $y^2 = -4x$

   Which are invariant properties under this transformation? Can you justify the name given to this transformation?

3. The transformation $T : \begin{cases} x = \dfrac{x' + y'}{2} \\ y = \dfrac{x' - y'}{2} \end{cases}$ is applied to the perpendicular lines $L_1 : 2x - 3y + 4 = 0$ and $L_2 : 3x + 2y - 6 = 0$. Determine whether the geometric property of perpendicularity is preserved under T

4. The set of affine transformations is one of the most fruitful of all types studied by mathematicians. They have the form

   $T \begin{cases} x = ax' + by' + c \\ y = dx' + ey' + f \end{cases}$. Many of the mappings studied in this chapter were special cases of this set. For example, the set of rotations are derived by letting the constants $a = \cos\theta$, $b = -\sin\theta$, $c = 0$, $d = \sin\theta$, $e = \cos\theta$ and $f = 0$.

   Consider the special case: $T \begin{cases} x = 2x' - 4y' + 1 \\ y = 3x' + 2y' - 4 \end{cases}$ and find its effect upon the graphs of the following:

   (a) $x^2 + y^2 = 4$

   (b) $4x^2 - 9y^2 = 36$

   (c) $4x - 3y + 12 = 0$

   (d) $4x - 3y - 1 = 0$

   (You probably cannot identify the images of (a) and (b) unless you study the Supplement to Chapter 7.)

5. In Problem 4, construct lines (c) and (d) and their images on the same set of coordinates. What tentative conclusion can you draw?

6. Prove that the mapping $(x,y) \longrightarrow (-x,-y)$ is a distance preserving transformation.

Table I

Natural Trigonometric Functions (Degree Measure)

Deg.	Sine	Cosine	Tangent	Cotangent	
0	0.000	1.000	0.000	*****	90
1	0.017	1.000	0.017	57.29	89
2	0.035	0.999	0.035	28.64	88
3	0.052	0.999	0.052	19.08	87
4	0.070	0.998	0.070	14.30	86
5	0.087	0.996	0.087	11.43	85
6	0.105	0.995	0.105	9.514	84
7	0.122	0.993	0.123	8.144	83
8	0.139	0.990	0.141	7.115	82
9	0.156	0.988	0.158	6.314	81
10	0.174	0.985	0.176	5.671	80
11	0.191	0.982	0.194	5.145	79
12	0.208	0.978	0.213	4.705	78
13	0.225	0.974	0.231	4.331	77
14	0.242	0.970	0.249	4.011	76
15	0.259	0.966	0.268	3.732	75
16	0.276	0.961	0.287	3.487	74
17	0.292	0.956	0.306	3.271	73
18	0.309	0.951	0.325	3.078	72
19	0.326	0.946	0.344	2.904	71
20	0.342	0.940	0.364	2.747	70
21	0.358	0.934	0.384	2.605	69
22	0.375	0.927	0.404	2.475	68
23	0.391	0.921	0.424	2.356	67
24	0.407	0.914	0.445	2.246	66
25	0.423	0.906	0.466	2.145	65
26	0.438	0.899	0.488	2.050	64
27	0.454	0.891	0.510	1.963	63
28	0.469	0.883	0.532	1.881	62
29	0.485	0.875	0.554	1.804	61
30	0.500	0.866	0.577	1.732	60
31	0.515	0.857	0.601	1.664	59
32	0.530	0.848	0.625	1.600	58
33	0.545	0.839	0.649	1.540	57
34	0.559	0.829	0.675	1.483	56
35	0.574	0.819	0.700	1.428	55
36	0.588	0.809	0.727	1.376	54
37	0.602	0.799	0.754	1.327	53
38	0.616	0.788	0.781	1.280	52
39	0.629	0.777	0.810	1.235	51
40	0.643	0.766	0.839	1.192	50
41	0.656	0.755	0.869	1.150	49
42	0.669	0.743	0.900	1.111	48
43	0.682	0.731	0.933	1.072	47
44	0.695	0.719	0.966	1.036	46
45	0.707	0.707	1.000	1.000	45
	Cosine	Sine	Cotangent	Tangent	Deg.

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.00	0.000	1.000	0.000	*****
.02	0.020	1.000	0.020	49.99
.04	0.040	0.999	0.040	24.99
.06	0.060	0.998	0.060	16.65
.08	0.080	0.997	0.080	12.47
.10	0.100	0.995	0.100	9.967
.12	0.120	0.993	0.121	8.293
.14	0.140	0.990	0.141	7.096
.16	0.159	0.987	0.161	6.197
.18	0.179	0.984	0.182	5.495
.20	0.199	0.980	0.203	4.933
.22	0.218	0.976	0.224	4.472
.24	0.238	0.971	0.245	4.086
.26	0.257	0.966	0.266	3.759
.28	0.276	0.961	0.288	3.478
.30	0.296	0.955	0.309	3.233
.32	0.315	0.949	0.331	3.018
.34	0.333	0.943	0.354	2.827
.36	0.352	0.936	0.376	2.657
.38	0.371	0.929	0.399	2.504
.40	0.389	0.921	0.423	2.365
.42	0.408	0.913	0.447	2.239
.44	0.426	0.905	0.471	2.124
.46	0.444	0.896	0.495	2.018
.48	0.462	0.887	0.521	1.921
.50	0.479	0.878	0.546	1.830
.52	0.497	0.868	0.573	1.747
.54	0.514	0.858	0.599	1.668
.56	0.531	0.847	0.627	1.595
.58	0.548	0.836	0.655	1.526
.60	0.565	0.825	0.684	1.462
.62	0.581	0.814	0.714	1.401
.64	0.597	0.802	0.745	1.343
.66	0.613	0.790	0.776	1.289
.68	0.629	0.778	0.809	1.237
.70	0.644	0.765	0.842	1.187
.72	0.659	0.752	0.877	1.140
.74	0.674	0.738	0.913	1.095
.76	0.689	0.725	0.950	1.052
.78	0.703	0.711	0.989	1.011
.80	0.717	0.697	1.030	0.971
.82	0.731	0.682	1.072	0.933
.84	0.745	0.667	1.116	0.896
.86	0.758	0.652	1.162	0.861
.88	0.771	0.637	1.210	0.827
.90	0.783	0.622	1.260	0.794

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.92	0.796	0.606	1.313	0.761
.94	0.808	0.590	1.369	0.730
.96	0.819	0.574	1.428	0.700
.98	0.830	0.557	1.491	0.671
1.00	0.841	0.540	1.557	0.642
1.02	0.852	0.523	1.628	0.614
1.04	0.862	0.506	1.704	0.587
1.06	0.872	0.489	1.784	0.560
1.08	0.882	0.471	1.871	0.534
1.10	0.891	0.454	1.965	0.509
1.12	0.900	0.436	2.066	0.484
1.14	0.909	0.418	2.176	0.460
1.16	0.917	0.399	2.296	0.436
1.18	0.925	0.381	2.427	0.412
1.20	0.932	0.362	2.572	0.389
1.22	0.939	0.344	2.733	0.366
1.24	0.946	0.325	2.912	0.343
1.26	0.952	0.306	3.113	0.321
1.28	0.958	0.287	3.341	0.299
1.30	0.964	0.268	3.602	0.278
1.32	0.969	0.248	3.903	0.256
1.34	0.973	0.229	4.256	0.235
1.36	0.978	0.209	4.673	0.214
1.38	0.982	0.190	5.177	0.193
1.40	0.985	0.170	5.798	0.172
1.42	0.989	0.150	6.581	0.152
1.44	0.991	0.130	7.602	0.132
1.46	0.994	0.111	8.989	0.111
1.48	0.996	0.091	10.98	0.091
1.50	0.997	0.071	14.10	0.071
1.52	0.999	0.051	19.67	0.051
1.54	1.000	0.031	32.46	0.031
1.56	1.000	0.011	92.62	0.011
1.58	1.000	-0.009	-108.65	-0.009
1.60	1.000	-0.029	-34.23	-0.029
1.62	0.999	-0.049	-20.31	-0.049
1.64	0.998	-0.069	-14.43	-0.069
1.66	0.996	-0.089	-11.18	-0.089
1.68	0.994	-0.109	-9.121	-0.110
1.70	0.992	-0.129	-7.697	-0.130
1.72	0.989	-0.149	-6.652	-0.150
1.74	0.986	-0.168	-5.853	-0.171
1.76	0.982	-0.188	-5.222	-0.191
1.78	0.978	-0.208	-4.710	-0.212
1.80	0.974	-0.227	-4.286	-0.233

The Greek Alphabet

Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gemma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ϵ	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	φ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

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For precisely defined analytic geometry terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

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